Adjoints

Initial/Final morphisms.

Let $R: \mathcal{C} \to \mathcal{D}$ be a functor. An initial morphism X to R is a morphism $\varphi: X \to R(A)$ such that for any other morphism $\psi: X \to R(B)$ there exists a unique morphism $f: A \to B$ such that $\psi = R(f)\varphi$.

$$\begin{tikzcd}
A \arrow[d, dashed, "f"] & & & R(A) \arrow[d, dashed, "R(f)"] \\
B & X \arrow[urr, "\varphi"] \arrow[drr, "\psi"'] & & R(B)
\end{tikzcd}$$

Dually, a final morphism from ~~X to R~~ R to X is a morphism $\varphi: R(A) \to X$ such that for all other morphisms $\psi: R(B) \to X$ there exists a unique morphism $f: B \to A$ s.t $\psi = \varphi \circ F(f)$

Suppose $R: \mathcal{C} \to \mathcal{D}$ is such that for all $X \in \mathcal{D}$, there exists an initial morphism from X to R. Then for all $f: X \to Y$ in $\mathcal{D}$, we have that there exists a unique map $g: A \to B$ such that

$$\begin{tikzcd}
A \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] & R(A) \arrow[d, "R(g)"] \\
B & Y \arrow[r] & R(B)
\end{tikzcd}$$

when the rows are the initial morphisms. We can define a functor $L: \mathcal{D} \to \mathcal{C}$ by $L(X) = A$ and $L(f) = g$.

where
- R is right adjoint to L
- L is left adjoint to R
- For every $A \in \mathcal{C}$, there exists a final morphism L to A. (✱)
- The initial morphisms $X \to RL(X)$ are the units η_X. (component of the unit)
- The final morphisms ~~R(A)~~ $LR(A) \to A$ are the counits ε_A. (comp. of the counit)

Dually, if $L: \mathcal{D} \to \mathcal{C}$ is a functor such that (✱) holds then it's a left adjoint.

It's a good exercise to go through why these are all equivalent. Note, the bijection is given by

$$\begin{aligned} \mathrm{Hom}(L(X), Y) &\longrightarrow \mathrm{Hom}(X, R(Y)) \\ f &\longmapsto R(f) \circ \eta_X \end{aligned}$$

Useful theorem to show a functor isn't left/right adjoint.

<u>Thm:</u> If F is a left adjoint, then $F(\varinjlim_i F_i) \simeq \varinjlim_i F(F_i)$

If F is a right adjoint, then $F(\varprojlim_i F_i) \simeq \varprojlim F(F_i)$.

ie left/right adjoints preserve co/limits.

Question 1:

Let $U : \underline{Grp} \to \underline{Set}$ be the forgetful functor.

We have a natural map $\varepsilon_x : FU(X) \to X$ given by the universal property of free groups:

$$\begin{tikzcd} FU(x) \arrow[dr, "\varepsilon_x"] & \\ & X \\ U(x) \arrow[uu, "i"] \arrow[ur, "id"'] & \end{tikzcd}$$

Moreover, this is a final morphism. Suppose we have another morphism $f : F(A) \to X$.
Then $f|_A : A \to X$ in $\underline{Set}$ and so we have a map $i \circ f|_A : A \to FU(X)$ and by universal property, we have that there exists a unique map that lifts this

$$\begin{tikzcd} F(A) \arrow[r, "g"] & FU(x) \\ A \arrow[u] \arrow[ur, "i\circ f|_A"'] & \end{tikzcd}$$

and so g is a unique morphism such that

$$\begin{tikzcd} F(A) \arrow[r, "g"] \arrow[dr, "f"'] & FU(x) \arrow[d, "\varepsilon_x"] \\ & X \end{tikzcd}$$ commutes.

Hence F is a left adjoint.
Observe $\cancel{F(\{1,2\} \times \{1,2\}) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}}$
$\cancel{F(\{1,2\}) \times}$

$F(\{*\} \times \{*\}) = \mathbb{Z}$, $F(\{*\}) \times F(\{*\}) = \mathbb{Z} \times \mathbb{Z}$.
Hence doesn't preserve limits and so is not a right adj.

Question 2

Observe by Yoneda that $\eta_X : X \to GF(X)$ is an isomorphism if and only if $\cancel{\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\eta_X \circ} \mathrm{Hom}_{\mathcal{C}}(X', GF}$

$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\eta_X \circ} \mathrm{Hom}_{\mathcal{C}}(X', GF(X))$ is a bijection for all $X' \in \mathcal{C}$.

Now, suppose F is fully faithful. Then we have

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow[\sim]{F\circ} \mathrm{Hom}_{\mathcal{D}}(F(X'), F(X))$$

$$\xrightarrow[G(-)\circ\eta_{X'}]{\sim} \mathrm{Hom}_{\mathcal{C}}(X', GF(X))$$

Since for $\varphi : X' \to X$, this maps to $GF(\varphi) \circ \eta_{X'} = \eta_X \circ \varphi$ since the following commutes

$$\begin{tikzcd} X' \arrow[r, "\eta_{X'}"] \arrow[d, "\varphi"'] & GF(X') \arrow[d, "GF(\varphi)"] \\ X \arrow[r, "\eta_X"'] & GF(X) \end{tikzcd}$$

Hence the map $\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\eta_X \circ} \mathrm{Hom}_{\mathcal{C}}(X', GF(X))$ is a bijection and η is an isomorphism by Yoneda.

Conversely, if η is an isomorphism, then as the following commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X', X) \arrow[r] \arrow[dr, "\sim", "\eta_{X*}"'] & \mathrm{Hom}_{\mathcal{D}}(F(X'), F(X)) \arrow[d, "\wr\ G(-)\circ\eta_{X'}"] \\
& \mathrm{Hom}(X', GF(X))
\end{tikzcd}$$

we must have the top row a bijection and so F is fully faithful.

Question: 3

(a) Let $X \in \mathcal{C}$ and consider the diagonal functor $\Delta : X \to X \times X$ given by the identity maps $X \xrightarrow{id} X$.

Suppose we have another morphism $\begin{pmatrix} f \\ g \end{pmatrix} : X \to Y \times Z$.

Here $\begin{pmatrix} f \\ g \end{pmatrix}$ is the unique morphism that makes the following commute:

$$\begin{tikzcd}
& X \arrow[d, "\binom{f}{g}"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
& Y \times Z \arrow[dl] \arrow[dr] & \\
Y & & Z
\end{tikzcd}$$

Now, let $p_1, p_2 : X \times X \to X$ be the projection maps.

Then $f \circ p_1$ and $f \circ p_2$ are maps $X \times X \to Y$ and $X \times X \to Z$ and so there exists a unique map $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} : X \times X \to Y \times Z$ that makes the following commute:

$$\begin{tikzcd}
 & X \times X \arrow[d, "\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}"] \arrow[ddl, bend right, "f \circ p_1"'] \arrow[ddr, bend left, "g \circ p_1"] & \\
 & Y \times Z \arrow[dl] \arrow[dr] & \\
Y & & Z
\end{tikzcd}$$

Moreover, the following commutes:

$$\begin{tikzcd}
X \arrow[r, "\Delta(\mathrm{id})"] \arrow[dr, "\begin{pmatrix} f \\ g \end{pmatrix}"'] & X \times X \arrow[d, "\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}"] \\
 & Y \times Z
\end{tikzcd}$$

and $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$ is the unique such map. Hence $X \to X \times X$ is an initial morphism and this works for all $X \in \mathcal{C}$. Hence F is a right adjoint.

b) Since <u>Ab</u> has finite biproducts, the dual argument to above shows that F is also a left adjoint.