Group Actions:

A group action of G on a set A is a map from $G \times A \to A$ satisfying

1) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$, and
2) $1 \cdot a = a$ for all $a \in A$.

- It is not hard to see that each $g \in G$ acts on A as a permutation of the elements. So an equivalent way to define a group action is as a group homomorphism $\varphi : G \to \mathrm{Sym}(A)$
- for $a \in A$, the stabalizer of a in G is the subgroup $G_a := \{ g \in G \mid g \cdot a = a \}$. (warm up)
- $G \cdot a = \{ g \cdot a \mid g \in G \}$ is the orbit of a under G. This is an equivalence class. (warm up)

Question 1

<u>G_a is a group:</u> $e \in G_a$ so nonempty. let $g, h \in G_a$. then $(gh^{-1}) \cdot a = gh^{-1}(h \cdot a) = g \cdot a = a$. Hence $gh^{-1} \in G_a$ and so subgroup.

<u>$G \cdot a$ is an equivalence class</u> in particular, we can define an equivalence relation on A by: $a \sim b \iff$ exists $g \in G$ s.t $g \cdot a = b$.

<u>reflexive</u>: clear as $1 \cdot a = a$

<u>symmetric</u>: if $a \sim b \iff g \cdot a = b \iff a = g^{-1} b \iff b \sim a$

<u>transitive</u>: if $a \sim b$ and $b \sim c$ then $\exists\, g, h \in G$ s.t $ga = b$, $hb = c$ so $hga = hb = c \iff a \sim c$.

Question 2

Proof: Let us define a mapping $\varphi: G \to G\cdot x$ given by $g \mapsto g\cdot x$. by def^n, this is surjective. Now, given $g, h \in G$, we have that

$$\begin{aligned} \varphi(g) = \varphi(h) &\iff g\cdot x = h\cdot x \\ &\iff h^{-1}g\cdot x = x \\ &\iff h^{-1}g \in G_x . \end{aligned}$$

ie, $\varphi(g) = \varphi(h)$ if and only if g and h are in the same left coset of G_x. Hence we get a bijective mapping $\varphi: G/G_x \to G\cdot x$
Hence it follows that $|G|/|G_x| = |G\cdot x|$.

Orbit stabaliser forms the basis of a lot of important results, we will go through a few of these

Question 3

We have that $\sum_{g\in G} |X^g| = |\{(g,x) \in G\times X \mid g\cdot x = x\}| = \sum_{x\in X} |G_x|$

and so by orbit-stabaliser

$$\sum_{g\in G} |X^g| = \sum_{x\in X} |G|/|G\cdot x| = |G| \sum_{x\in X} \frac{1}{|G\cdot x|} = |G||X/G|$$

Question 4

note $C_G(x_i) =$ centralizer of x_i in $G = \{g\in G \mid g x_i g^{-1} = x_i\}$.
G acts on itself via conjugation. ie $g\cdot h := ghg^{-1}$. Under this action, $C_G(x_i)$ is the stabalizer subgroup of x_i, and so by orbit-stabaliser $|G : C_G(x_i)| =$ size of the orbit of x_i under G. Hence, as orbits

$|G : C_G(x_i)| =$ size of the orbit of x_i under G. Hence, as orbits partition the set, the result follows.

Question 5

From orbit-stabaliser, as $|G| = |G \cdot x| \, |G_x|$, we see that orbit sizes must divide the order of the group. Since $x \in X^G$ are precisely the elements with trivial orbits, the result follows.

Question 6

Let $X = \{ (g_1, \dots, g_p) \in G^p \mid g_1 \cdots g_p = e \}$. Observe that we have the first $p-1$ entries freely determined out of G and then g_p fixed. Hence $|X| = |G|^{p-1}$. Now, consider the action $C_p \curvearrowright X$ via permuting entries. Now, the orbits under this have either size 1 or p (from orbit-stabaliser). From the previous question, we see that $|X^G| \equiv 0 \pmod{p}$ and as $|X^G| \neq 0$ since $(e, e, \dots, e) \in X$ and is fixed under the action, we see there must be an element $g \in G$ s.t. $(g, \dots, g) \in X$. Hence $g^p = e$ $\blacksquare$