MATH 210C: HOMEWORK 8

Problem 76. Compute the character table for C_5 over $\mathbb{C}$. Verify that the simple characters for C_5 over $\mathbb{C}$ are orthogonal. (Recall Problem 72.)

Problem 77. Let S_3 denote the symmetric group on 3 elements.

(a) Show that S_3 has three conjugacy classes.
(b) Show that S_3/A_3 has two characters of dimension 1.
(c) Show that S_3 has an irreducible character of dimension 2. Prove that we can realize this 2-dimensional representation as a k-vector space generated by the roots of a cubic polynomial with Galois group S_3 over any base field k (which remains irreducible after tensoring with the algebraic closure of k).
(d) Write down an idempotent for each of the simple components of $\mathbb{C}[S_3]$. What is the multiplicity of each irreducible representation of S_3 in the regular representation?

Problem 78. Compute the character table for S_3 over $\mathbb{C}$.

Problem 79. Let S_4 denote the symmetric group on 4 elements.

(a) Show that S_4 has five conjugacy classes.
(b) Show that A_4 has a unique non-cyclic order 4 subgroup N such that $N \triangleleft S_4$ and $S_4/N \cong S_3$. Hence three irreducible characters of S_4 you can compute using Problem 78.
(c) Determine the dimensions of the other two irreducible characters.
(d) Using an irreducible quartic polynomial with Galois group S_4, employ a method similar Problem 77(c) to find one of the remaining irreducible representations. Denote it ρ.
(e) Define a representation ρ' by

$$\rho'(\sigma) = \mathrm{sgn}(\sigma)\rho(\sigma)$$

where $\mathrm{sgn} : S_4 \to \{\pm 1\}$ is the signature group homomorphism with $\ker(\mathrm{sgn}) = A_4$. Prove that ρ' is also irreducible but not isomorphic to ρ, therefore completing the characterisation.
(f) Show that an irreducible 3-dimensional representation of S_4 is still irreducible on A_4. What other irreducible representations of A_4 are there?

Problem 80. Compute the character tables for A_4 and S_4 over $\mathbb{C}$.

Problem 81. Let Q_8 denote the quaternion group

$$Q_8 = \{\pm 1, \pm a, \pm b, \pm c\} \text{ with } a^2 = b^2 = c^2 = abc = -1$$

(a) Show that Q has five conjugacy classes.
(b) Let $N = \{\pm 1\}$. Show that N is normal and Q_8/N has four irreducible characters of dimension 1.
(c) Prove that Q_8 has a 2-dimensional irreducible character $\rho : Q_8 \to \mathrm{GL}_2(\mathbb{C})$. Compute $\rho(b)$ and $\rho(c)$ if we let

$$\rho(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

(d) Let $\mathbb{H}$ be the 4-dimensional $\mathbb{R}$-algebra on $\{1, a, b, c\}$ with relations as above. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$. Relate this to (c).

Problem 82. Compute the character table for Q_8 over $\mathbb{C}$.

Problem 83. Let D_{2n} denote the dihedral group of the regular n-gon, or in terms of generators and relations,

$$D_{2n} = \langle \sigma, \tau : \sigma^n = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

Assume that n is even for this problem.

(a) Show that D_{2n} has four representations of dimension 1 given by the four set functions $\{\sigma, \tau\} \to \{-1, 1\}$.
(b) Let $C_n = \langle \sigma \rangle$ be the cyclic subgroup of D_{2n}. Let $\psi_r : C_n \to \mathbb{C}^\times$ be the characters given by $\psi_r(\sigma) = \zeta^r$ for a fixed primitive nth root of unity ζ and $r \in \{0, \ldots, n-1\}$. Let χ_r be the induced representations on D_{2n}. Prove that $\chi_r = \chi_{n-r}$.
(c) For $0 < r < n/2$, show that χ_r is an irreducible character of dimension 2, so that we get $n/2 - 1$ distinct irreducible characters of dimension 2.
(d) Show that (a) and (c) characterise all irreducible characters of D_{2n}.