MATH 210C: HOMEWORK 3

Problem 20. Let F be a field and let $K = F(T)$.

(a) Show that $T \mapsto T^{-1}$ defines an automorphism of K.
(b) Transport the valuation at zero (i.e. at T) by this automorphism.
(c) Describe the above 'valuation at infinity' explicitly on a fraction of polynomials.
(d) Verify directly that the explicit valuation obtained under (c) is a discrete valuation.

Problem 21. Describe the p-valuation on the field of p-adic numbers $\mathbb{Q}_p$.

Problem 22. Show that if p and q are distinct prime numbers, then $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic.

Problem 23. Let A be a noetherian domain and $\mathfrak{p}$ a prime ideal which is principal.

(a) Show that $\mathfrak{p}$ has height one, that is, there is no prime ideal strictly between (0) and $\mathfrak{p}$.
(b) Give an example of a domain A and a prime of height one which is not principal.

Problem 24. Prove that there exists a (non-noetherian) local domain A whose maximal ideal $\mathfrak{m}$ is principal, yet A has Krull dimension more than one.

Problem 25. Show that localization commutes with integral closure: let A be a domain and K its field of fractions. Let $S \subset A$ be a multiplicative subset. Show that $\overline{S^{-1}A} = S^{-1}\overline{A}$.

Problem 26. Give an example of a domain A of Krull dimension one that is integrally closed but not Dedekind.

Problem 27. Let R be a commutative ring and M an R-module. Let $M^\vee = \mathrm{Hom}_R(M, R)$ be its dual and $\varepsilon : M \otimes M^\vee \to R$ the evaluation morphism.

(a) Show that if R is a DVR and M is a nonzero ideal, then ε is an isomorphism.
(b) Show that the same result is true if R is a Dedekind domain.
(c) For R noetherian and M finitely generated (or in general for M finitely presented) show that ε is an isomorphism if and only if M is locally free of rank one.

Problem 28. Let $A = \mathbb{R}[X,Y]/\langle X^2+Y^2-1\rangle = \mathbb{R}[x,y]$.

(a) Show that A is a Dedekind domain but not a PID.
(b) Let $M = \langle 1+x, y\rangle$. Show that the ideal M satisfies $A^2 \simeq M \oplus M$ via the homomorphism $(1,0) \mapsto (1+x, y)$ and $(0,1) \mapsto (-y, 1+x)$.
(c) Show that $M \otimes M^\vee \simeq A$ via the homomorphism $m_1 \otimes m_2 \mapsto \frac{m_1 m_2}{2(1+x)}$. (Show that this homomorphism is well-defined!)
(d) What is the relationship between M and the Möbius band?

Problem 29. Let A be a Dedekind domain. Define a *fractional ideal* to be a nonzero A-submodule $\mathfrak{a}$ of $K = \operatorname{Frac} A$ such that there exists a nonzero $c \in A$ such that $c\mathfrak{a} \subset A$.

(a) Prove that every maximal ideal $\mathfrak{m}$ is invertible. That is, let
$$\mathfrak{m}^{-1} := \{x \in K : x\mathfrak{m} \subset A\}.$$
Then $\mathfrak{m}^{-1}\mathfrak{m} = A$.
(b) Prove that every non-zero ideal is invertible by a fractional ideal. Hint: suppose otherwise, and consider the (non-empty) family of non-invertible ideals. Use the fact that A is noetherian to reach a contradiction.
(c) Conclude that the set of fractional ideals is an abelian group under multiplication.

Problem 30. Let R be a noetherian ring. Let $\operatorname{Pic} R$, the Picard group of R, denote the set of isomorphism classes of invertible R-modules M as in Problem 27.

(a) Prove that $\operatorname{Pic} R$ is an abelian group under $\otimes$.
(b) If A is a Dedekind domain, prove that every fractional ideal is an invertible A-module, and prove that this association of fractional ideals into $\operatorname{Pic} A$ is a surjective group homomorphism.
(c) Compute the kernel of the above group homomorphism.