Math 210C Homework 8

due 5/30/2013

Problem 1 *(a) Show that there is no group G such that $\mathbb{C}G \cong M_2(\mathbb{C})$.*

(b) Is there a group G such that $\mathbb{C}G \cong \mathbb{C} \times M_2(\mathbb{C})$?

(c) Find equivalent conditions for $\mathbb{C}G$ to be isomorphic to a product of copies of $\mathbb{C}$.

(d) Describe $\mathbb{C}G$ for G of order 6.

Problem 2 *(a) Let G be a finite group and F an algebraically closed field of characteristic zero or p with p not dividing the order of G. Show that the number of degree 1 representations of G is the order of G^{ab}.*

(b) Show that G is abelian if and only if every irreducible character of G has degree 1.

Problem 3 *(a) Give a complete table of character values for each irreducible representation ρ of the group S_3.*

(b) Let V be a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V = 5$. The group S_3 acts naturally on the vector space $V \otimes V \otimes V$ by permutation of the three tensor factors. For each irreducible representation L of S_3, find the multiplicity $[V \otimes V \otimes V : L]$ of L in $V \otimes V \otimes V$.

Problem 4 *Consider S_4.*

(a) Show that there are 5 conjugacy classes.

(b) Recall that A_4 has a unique normal subgroup of order 4, which we denote by N. Show that $S_4/N \cong S_3$. Conclude that the representations of S_3 give rise to representations of S_4.

(c) Conclude that S_4 has only two other irreducible representations, each of dimension 3.

(d) Let $X^4 + a_2X^2 + a_1X + a_0$ be an irreducible polynomial over a field k with Galois group S_4. Show the roots generate a 3-dimensional k-vector space V, and

that the representation of S_4 on this space is irreducible. This gives us one of the missing representations, which we call ρ.

(e) Define ρ' to be $\rho'(\sigma) = \rho(\sigma)$ if ρ is even and $\rho'(\sigma) = -\rho(\sigma)$ if σ is odd. Show that ρ' is irreducible, remains irreducible after tensoring with $\bar{k}$, and is not isomorphic to ρ.

(f) Use the above work to determine the character table for S_4.

Problem 5 *Now consider $A_4 \subset S_4$.*

(a) Show that the 3-dimensional representations of the previous question provide an irreducible representation of A_4.

(b) Show that all other irreducible representations of A_4 are 1-dimensional.

(c) Determine the character table for A_4.

Problem 6 *Using the character tables for cyclic groups, calculate the character tables for each of the three abelian groups of order 8.*

Problem 7 *Consider the dihedral group $D_n = \langle x, y \,|\, x^2 = 1,\ y^n = 1,\ xyx = y^{-1} \rangle$ with $2n$ elements.*

(a) Let $\epsilon = e^{2\pi i/n}$, and for $1 \leq j < n/2$, let $A_j = \begin{pmatrix} \epsilon^j & 0 \\ 0 & \epsilon^{-j} \end{pmatrix}$ and $B_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that $\rho_j : D_n \to GL(2\mathbb{C})$, $y^r x^s \mapsto (A_j)^r (B_j)^s)$ gives an irreducible representation of G. Show that ρ_i and ρ_j are not equivalent for $i \neq j$.

(b) For n odd, complete the character table by lifting the irreducible characters of $D_n/\langle y \rangle$ to D_n.

(c) For n even, complete the character table by lifting the irreducible characters of $D_n/\langle y^2 \rangle$ to D_n.