Math 210C Homework 6

due 5/16/2013

Problem 1 *(Jacobson Density Theorem) Let M be a semi-simple A-module and let $B = End_A(M)$.*

(a) Show that M is a left B-module and define $A \to End_B(M)$.

Let $f \in End_B(M)$. Let $x_1, ..., x_n \in M$.

(b) Show there is $a \in A$ such that $f(x_i) = ax_i$ for $i = 1, ..., n$.

(c) Show that if M is finitely generated over B, then $A \to End_B(M)$ is surjective.

Problem 2 *Let A be a ring and $a, b \in A$. Show that the element $1 - ab$ is invertible if and only if the element $1 - ba$ is invertible.*

Problem 3 *Show that for any ring A, one has:*

(a) $Rad(M_n(A)) = M_n(Rad(A))$.

(b) $Rad(A/Rad(A)) = 0$.

Problem 4 *For $n \geq 1$, find the Jacobson radical of the ring of upper triangular $n \times n$-matrices over a field k.*

Problem 5 *Give an example where the Jacobson Radical is not the intersection of all two-sided maximal ideals.*

Problem 6 *Let A be an artinian ring (e.g. a finite dimensional algebra). Show that $Rad(A)^n = 0$ for some large enough n.*

Problem 7 *Let $A \subset \mathbb{Q}$ be the subgroup formed by all rational numbers of the form p/q, where p, q are integers such that $\gcd(p, q) = 1$ and q is odd. Let B be the ring of 2×2-matrices of the form $\begin{pmatrix} a & u \\ 0 & v \end{pmatrix}$, $a \in A$, $u, v \in \mathbb{Q}$. Find $Rad(A)$ and show that $Rad(B) \subset \begin{pmatrix} Rad(A) & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. Conclude that $\bigcap_{n \geq 1} Rad(B)^n \neq 0$.*

Problem 8 *Show that every simple module is indecomposable. Give an example of an indecomposable module which is not simple.*

Problem 9 *Show that M is indecomposable if and only if its endomorphism ring has no idempotents other than 0 and 1.*

Problem 10 *In the category of A-modules, define*

$rad_A(X,Y) = \{f \in Hom_A(X,Y) \mid 1_X - gf$ *is invertible in* $End_A(X)$ *for all* $g \in Hom_A(Y,X)\}$

for X and Y A-modules. The Jacobson radical of the category A-Mod is

$rad(A\text{-}Mod) = \bigcup_{X,Y \in A\text{-}Mod} rad_A(X,Y)$. *Show that the Jacobson radical is an ideal in A-Mod.*