Math 210C Homework 5

due 5/9/2013

1. Let D be a division ring and $n \geq 1$ and let $M_r(D)$ act on D^r as on column vectors.

(a) Prove that D^r is a simple $M_r(D)$-module.

(b) Prove that any $M_r(D)$-module is a direct sum of copies of D^r.

2. Give an example of a module that does not have a simple submodule.

3. Show that a commutative semisimple ring is a finite direct product of fields.

4. Let A be a ring and $n \geq 1$.

(a) Determine all two-sided ideals of $M_n(A)$.

(b) Show that if A is simple then so is $M_n(A)$.

5. Give an example of a simple ring R which is not simple as a left R-module. Identify the simple rings R which are simple as left R-modules.

6. The Weyl algebra $W = K[X, Y]$ (as an abelian group) but with $YX = XY+1$ (or equivalently, $W = K\langle X, Y\rangle / \langle YX - XY - 1\rangle$) is simple but not semi-simple (not aritinian).

7. Let R be a semisimple ring with minimal nonisomorphic left ideals $A_1, ..., A_n$ and $B_1, ..., B_n$ its simple components where $B_i = \sum_{A \cong A_i} A$. Show if $0 \neq M$ is an R-module then $B_i M$ is a sum of irreducible submodules of M all isomorphic to A_i and further that $M = \bigoplus B_i M$.

8. Find all finite-dimensional simple algebras over $\mathbb{R}$.

9. Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ as $\mathbb{C}$-algebras, where $\mathbb{H}$ is the real quaternion algebra. (Defined in class.)