Math 210C Homework 4

due 5/2/2013

1. Show that the p-adic integers $\mathbb{Z}_p^\wedge$ form a DVR. Describe the discrete valuation.

2. Show that a Dedekind domain with finite spectrum is a PID.

3. Show that a Dedekind domain is a PID if and only if it is a UFD.

4. Let I be a nonzero fractional ideal in the Dedekind domain R. Write I as a direct summand of a free R-module to show that I is projective. (Suggestion: $I \oplus I^{-1}$)

5. Let $\mathcal{O}$ be the ring of integers in the algebraic closure $\mathcal{A}$ of $\mathbb{Q}$.

(a) Show that the infinite sequence of ideals in $\mathcal{O}$,

$(2) \subset (\sqrt{2}) \subset (\sqrt[4]{2}) \subset (\sqrt[8]{2}) \subset \cdots$,

is strictly increasing, and so $\mathcal{O}$ is not Noetherian.

(b) Show that $\mathcal{O}$ has Krull dimension 1.

6. Let $A = \mathbb{R}[X,Y]/(X^2 + y^2 - 1) = \mathbb{R}[x,y]$. Let $I = \langle x, 1-y \rangle \subset A$.

(a) Show that A is a Dedekind domain.

(b) Show that I is projective (of rank 1) but not free.

(c) Show that $I \oplus I \simeq A^2$.

7. (a) Describe all simple $\mathbb{Z}$-modules.

(b) Which finitely generated $\mathbb{Z}$-modules are not semi-simple?

8. Let $I \subset R$ be a left ideal.

(a) Show that I is simple as a left R-module if and only if it is minimal as a left ideal.

(b) Show that the module R/I is simple if and only if I is a maximal left ideal