Math 210C Homework 2

due 4/18/2013

1. (a) Prove that a UFD is integrally closed.

Let k be a field, and consider the ring $A = k[X,Y]/(Y^2 - X^3)$.

(b) Prove that A is not integrally closed, hence is not a UFD.

(c) Show that for the inclusion
$k[T^2, T^3] \hookrightarrow k[T]$, the associated map $\mathrm{Spec} k[T] \to \mathrm{Spec} k[T^2, T^3]$ is a bijection.

2. (Going Down Theorem) Let B be an integral domain, $A \subset B$ a subring with B integral over A and A integrally closed.

(a) Let $P \in \mathrm{Spec}(A)$ and $\alpha \in PB$. Let K be the field of fractions of A and h the minimal polynomial of α over K. Then $h = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X + a_n$ with $a_i \in P$ and $n > 0$. (Suggestion: write $\alpha = \sum_{i=1}^n p_i b_i$, $p_i \in P$ and $b_i \in B$. Replace B with $B' = A[b_1, ..., b_n]$. Let $L = K(b_1, ..., b_n)$. Let h_i be the minimal polynomial of b_i over K. Let M be a splitting field of $hh_1 \cdots h_n$ over L (hence over K). Let R be the integral closure of A in M. The roots α_i of h in a splitting field are integral over A, hence so are the a_i. Show $\alpha_i \in PR$, so $a_i \in PR$. Then show $a_i \in P$.)

(b) Let $P_1, P_2 \in \mathrm{Spec}(A)$ with $P_1 \subsetneq P_2$ and $Q_2 \in \mathrm{Spec}(B)$ with $Q_2 \cap A = P_2$. Show that $P_1 B \cap S = \emptyset$, where $S = \{uv : u \in A - P_1,\, v \in B - Q_2\}$. Conclude that there is $Q_1 \in \mathrm{Spec} B$ such that $Q_1 \subset Q_2$ and $Q_1 \cap A = P_1$. (Suggestion: consider the minimal polynomial of uv, which gives a polynomial in v (which is integral over A). Consider this mod P_1 to show that $v \in \sqrt{BP_1}$)

3. Prove that the ring of Gaussian integers, $\mathbb{Z}[i]$, is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(i)$.

4. Let R be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ for d a square free integer.

(a) If $d \equiv 2, 3 \bmod 4$, prove that $R = \mathbb{Z}[\sqrt{d}]$ and $\{1, \sqrt{d}\}$ is an integral basis.

(b) If $d \equiv 1 \bmod 4$, prove that $R = \mathbb{Z}[\frac{-1+\sqrt{d}}{2}]$ and $\{1, \frac{-1+\sqrt{d}}{2}\}$ is an integral basis.

5. Show that a noetherian ring has only finitely many minimal primes.

6. If R is an integral domain with field of fractions F, show that F is a finitely generated R-module if and only if $R = F$.