Math 210C Homework 1

due 4/11/2013

1. Let R be a commutative ring.

(a) Show that, for a commutative ring R, $\mathrm{Spec}(R)$ is a topological space with closed sets given by $V(I) = \{P \in \mathrm{Spec}(R) : I \subset P\}$ for ideals $I \subset R$.

(b) Show that the basic open sets $D(f) = \{P \in \mathrm{Spec}(R) : f \notin P\}$ (for $f \in R$) form a basis for the Zariski topology.

(c) Give an example where $\mathrm{Spec}(R)$ is not Hausdorff. Give an example where $\mathrm{Spec}(R)$ is Hausdorff.

2. (a) Describe the topological space $\mathrm{Spec}(\mathbb{Z})$.

(b) Describe the topological space $\mathrm{Spec}(\mathbb{Z}[X])$.

3. Let $\phi \in M_n(\mathbb{C})$. Let A be the commutative subring of $M_n(\mathbb{C})$ over $\mathbb{C}$ generated by ϕ. That is, $A = \{a_0 + a_1\phi + a_2\phi^2 + \cdots + a_n\phi^n \,|\, n \geq 0,\, a_i \in \mathbb{C}\}$. Let $f(X)$ be the minimal polynomial of ϕ. Prove that we have an isomorphism $\mathbb{C}[X]/(f(X)) \to A$ of rings over $\mathbb{C}$. Conclude that there is a bijection between $\mathrm{Spec}(A)$ and the set of eigenvalues of ϕ.

4. (a) Let $\phi : A \to B$ be a homomorphism of commutative rings. Define a continuous function $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

(b) If R is a commutative ring, $S \subset R$ a multiplicatively closed subset, describe $\mathrm{Spec}(S^{-1}R)$.

5. The *radical* of an ideal I in a commutative ring A is defined by $\sqrt{I} = \{x \in A \,|\, \exists n \text{ such that } x^n \in I\}$. The *Jacobson radical* of A is $\mathrm{Rad}(A) = \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} \mathfrak{m}$.

(a) Show that $\sqrt{I} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A),\, I \subset \mathfrak{p}} \mathfrak{p}$.

(b) Show $\sqrt{0} \subset \mathrm{Rad}(A)$. Give an example where equality does not hold.

6. Let R be a commutative ring, M a finitely generated R-module, and $I \subset R$ an ideal.

(a) Show that if $M \otimes_R R/I = 0$ and $I \subset \mathrm{Rad}(R)$, then $M = 0$.

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

(b) Show that $k \otimes_R M = 0$ implies $M = 0$.

(c) Let $x_1, ..., x_m \in M$ be elements whose images $\bar{x}_1, ..., \bar{x}_m \in M/\mathfrak{m}M$ form a basis for the k-vector space $M/\mathfrak{m}M$. Show that $x_1, ..., x_m$ generate M as an R-module.

7. (Prime Avoidance) Let R be a commutative ring, and let $\mathfrak{p}_1, ..., \mathfrak{p}_m \subset R$ be ideals, at most two of which are not prime. Suppose I is an ideal such that $I \subset \bigcup_{i=1}^m \mathfrak{p}_i$. Prove that $I \subset \mathfrak{p}_k$ for some k.

8. (a) Prove that a UFD is integrally closed.

Let k be a field, and consider the ring $A = k[X, Y]/(Y^2 - X^3)$.

(b) Prove that A is not integrally closed, hence is not a UFD.

(c) Show that for the inclusion $k[T^2, T^3] \hookrightarrow k[T]$, the associated map $\mathrm{Spec}k[T] \to \mathrm{Spec}k[T^2, T^3]$ is a bijection.

9. (Going Down Theorem) Let B be an integral domain, $A \subset B$ a subring with B integral over A and A integrally closed.

(a) Let $P \in \mathrm{Spec}(A)$ and $\alpha \in PB$. Let K be the field of fractions of A and h the minimal polynomial of α over K. Then $h = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X + a_n$ with $a_i \in P$ and $n > 0$. (Suggestion: write $\alpha = \sum_{i=1}^n p_i b_i$, $p_i \in P$ and $b_i \in B$. Replace B with $B' = A[b_1, ..., b_n]$. Let $L = K(b_1, ..., b_n)$. Let h_i be the minimal polynomial of b_i over K. Let M be a splitting field of $hh_1 \cdots h_n$ over L (hence over K). Let R be the integral closure of A in M. The roots α_i of h in a splitting field are integral over A, hence so are the a_i. Show $\alpha_i \in PR$, so $a_i \in PR$. Then show $a_i \in P$.)

(c) Let $P_1, P_2 \in \mathrm{Spec}(A)$ with $P_1 \subsetneq P_2$ and $Q_2 \in \mathrm{Spec}(B)$ with $Q_2 \cap A = P_2$. Show that $P_1 B \cap S = \emptyset$, where $S = \{uv : u \in A - P_1, v \in B - Q_2\}$. Conclude that there is $Q_1 \in \mathrm{Spec}B$ such that $Q_1 \subset Q_2$ and $Q_1 \cap A = P_1$.

10. Let $d \in \mathbb{Z}$. Show that $\mathbb{Z} \subset \mathbb{Z}[\sqrt{d}]$ is an integral extension. Give examples of primes in $\mathrm{Spec}(\mathbb{Z})$ which have multiple preimages in $\mathrm{Spec}(\mathbb{Z}[\sqrt{d}])$, for explicit values of d.