Math 210C Homework 9

Question 1. Suppose L/K is a non-separable field extension. Recall that this implies that the characteristic is $p > 0$ and that there exists an subextension $K \subset L' \subset L$, where L'/K is separable and L/L' is purely inseparable. Show that there exists a number $u > 0$ such that $N_{L/K}(x) = N_{L'/K}(x)^{p^u}$, and that $\mathrm{Tr}_{L/K}(x) = 0$, for all $x \in L$.

Question 2. Prove that if $M/L/K$ is a tower of finite field extensions then we have:

(a) $N_{M/K} = N_{L/K} \circ N_{M/L}$.

(b) $\mathrm{Tr}_{M/K} = \mathrm{Tr}_{L/K} \circ \mathrm{Tr}_{M/L}$.

Question 3. Decide whether the following polynomials are solvable by radicals.

(a) $X^5 - 6X + 3$.

(b) $X^5 - 5X + 12$.

(c) $X^6 - 4X^3 + 1$.

(d) $X^6 + 24X - 20$.

Question 4 (Criterion for the Solvability of a Quintic)**.** From Problem 7, we see that a quintic polynomial is solvable if and only if its Galois group is contained in the Frobenius group of order 20. Consider the quintic $f(X) = X^5 + AX + B$. Then it is known that the Galois group is contained in the Frobenius group of order 20 if and only if the following degree 6 polynomial $g(X)$ has a rational root:

$$\begin{aligned} g(X) = X^6 + 8AX^5 + 40A^2X^4 + 160A^3X^3 + 400A^4X^2 \\ +(512A^5 - 3125B^4)X - 9375AB^4 + 256A^6. \end{aligned} \tag{1}$$

This reduces the problem of deciding if a quintic is solvable by radicals to a use of the rational root test. Use this test to determine the solvability by radicals for the degree 5 polynomials in Problem 3, as well as the polynomials $X^5 - X - 1$, $X^5 + X - 1$, and $X^5 - 2X + 5$.

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Question 5. Let V be finite dimensional vector space over F and consider an F-bilinear pairing $\beta : V \times V \longrightarrow F$. Show that this pairing is non-degenerate (that is, for $v \in V$, if $\beta(v, w) = 0\ \forall w \in V$ then $v = 0$) if and only if the induced morphism $V \to V^\vee$ is an isomorphism ($V^\vee = \mathrm{Hom}_F(V, F)$ is the dual of V).

Question 6. Let L/K be a separable finite extension. When is $\mathrm{Tr}_{L/K}(1) = 0$? Show however that $\mathrm{Tr}_{L/K}$ is non-zero. Deduce that the pairing $(x, y) \mapsto \mathrm{Tr}(xy)$ is non-degenerate.

Question 7. Let p be prime. Show that any solvable subgroup of S_p of order divisible by p is contained in the normalizer of a Sylow p-subgroup of S_p (which turns out to be a Frobenius group of order $p(p-1)$). Conclude that an irreducible polynomial $f(X) \in \mathbb{Q}[X]$ of degree p is solvable by radicals if and only if its Galois group is contained in the Frobenius group of order $p(p-1)$.

[Hint: Let $G \leq S_p$ be a solvable subgroup of order divisible by p. Then G contains a p-cycle, and hence is transitive on the set $\{1, \ldots, p\}$. Let $H < G$ be the stabilizer in G of the element 1, so H has index p in G. Show that H contains no non-trivial normal subgroups of G. Let $G^{(n-1)}$ be the last non-trivial subgroup in the derived series for G. Show that $H \cap G^{(n-1)} = 1$ and conclude that $|G^{(n-1)}| = p$, so that the Sylow p-subgroup of G (which is also a Sylow p-subgroup of S_p) is normal in G.]

Question 8. Let $F = \mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/5}$, and let E/F be a cyclic Galois extension of degree 5. Prove that there exists an $\alpha \in F$ such that $E = F(\sqrt[5]{\alpha})$. (Hint: Find $\beta \in E$ such that $\sigma(\beta) = \zeta\beta$, where σ is a generator for $\mathrm{Gal}(E/F)$.)