Math 210C Homework 8

Question 1. Determine the character table for S_3.

Question 2. Consider S_4.

(a) Show that there are 5 conjugacy classes.

(b) Recall that A_4 has a unique normal subgroup of order 4, which we denote by N. Show that $S_4/N \cong S_3$. Conclude that the representations of S_3 determined in Homework 7 give rise to representations of S_4.

(c) Conclude that S_4 has only two other irreducible representations, each of dimension 3.

(d) Let $X^4 + a_2X^2 + a_1X + a_0$ be an irreducible polynomial over a field k with Galois group S_4. Show the roots generate a 3-dimensional k-vector space V, and that the representation of S_4 on this space is irreducible. This gives us one of the missing representations, which we call ρ.

(e) Define ρ' to be $\rho'(\sigma) = \rho(\sigma)$ if σ is even and $\rho'(\sigma) = -\rho(\sigma)$ if σ is odd. Show that ρ' is irreducible, remains irreducible after tensoring with $\overline{k}$, and is not isomorphic to ρ.

(f) Use the above work to determine the character table for S_4.

Question 3. Now consider $A_4 \subset S_4$.

(a) Show that the 3-dimensional representations of Question 2 provide an irreducible representation of A_4.

(b) Show that all other irreducible representations of A_4 are 1-dimensional.

(c) Determine the character table for A_4.

Question 4. Determine the character tables for D_4.

* * *

Question 5. The work of Question 4 can be generalized to D_n where n is any even integer. (For convenience, recall that D_n is generated by σ and τ with $\sigma^n = \tau^2 = 1$ and $\tau\sigma\tau = \sigma^{-1}$.)

(a) Show that there are four irreducible representations of dimension 1.

(b) Let $C_n \subset D_n$ be the cyclic subgroup of order n, generated by $\sigma \in D_n$. For each integer $r = 0, 1, \ldots, n-1$, let ψ_r be the representation given by $\psi_r(\sigma) = \zeta_n^r$. Let χ_r be the induced character for D_n. Show that $\chi_r = \chi_{n-r}$.

(c) Show that, for $0 < r < \frac{n}{2}$, χ_r is simple of dimension 2, and so one gets $(\frac{n}{2} - 1)$ distinct characters of dimension 2.

(d) Prove that the simple characters in parts (a) and (c) are all the simple characters for D_n. Compare with the results of Question 4.

Question 6. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the usual quaternion group.

(a) Show that Q has 5 conjugacy classes.

(b) Let $A = \{\pm 1\}$. Show that Q/A (and hence Q) has 4 simple characters.

(c) Show that there is only one more simple character, of dimension 2. Find the matrices of the corresponding representation.

Question 7. Use the work from Question 6 to determine the character table for Q.

Question 8. Using the character tables for cyclic groups, calculate the character tables for each of the 3 abelian groups of order 8.