Math 210C Homework 7

Question 1. Let k be an algebraically closed field. Let D be a finite dimensional division k-algebra. Show that $D = k$. [Hint: Pick $x \in D$ and consider the subring $k[x] \subset D$.] What can you deduce about finite dimensional semi-simple k-algebras?

Question 2. Let p be a prime and C_p be the cyclic group of order p. Describe all finitely generated $\mathbb{F}_p C_p$-modules.

Question 3. Let G be a p-group and k be a field of characteristic p. Show that kG is a local ring. [Hint: Show that the augmentation ideal $I := \ker(kG \xrightarrow{\epsilon} G)$ is nilpotent.]

Question 4. Consider S_3 the permutation group on 3 letters. Prove:

(a) The signature defines a one-dimensional representation of S_3 over $\mathbb{Q}$.

(b) S_3 acts on the 2-dimensional $\mathbb{Q}$-vector space $M = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 \mid x_1 + x_2 + x_3 = 0\}$.

(c) M is an irreducible $\mathbb{Q}S_3$-module.

(d) Give all irreducible representations of S_3 over $\mathbb{Q}$.

(e) $\mathbb{Q}S_3 \simeq \mathbb{Q} \times \mathbb{Q} \times \mathrm{M}_2(\mathbb{Q})$.

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Question 5. Let A be a ring. An *involution* on A is a ring homomorphism $\tau : A^{\mathrm{op}} \to A$ such that $\tau^2 = \mathrm{id}$.

(a) Show that an involution is the same as an abelian group isomorphism $\tau : A \xrightarrow{\sim} A$ such that $\tau(ab) = \tau(b)\tau(a)$ and $\tau^2 = 1$.

(b) Show that the identity $A \to A$ is an involution if and only if A is commutative.

(c) Let G be a group and R be commutative. Show that the group algebra $A = RG$ has a unique R-linear involution such that $\tau(g) = g^{-1}$ for all $g \in G$.

(d) Let M be a left A-module. Recall the natural structure of right A-module on $M^* := \mathrm{Hom}_A(M, A)$.

(e) Let A be a ring with involution. Show that the category of left A-modules is isomorphic to the category of right A-modules.

(f) Let A be a ring with involution. Show that $M \mapsto M^*$ induces a functor from A–Mod to itself.

(g) Let M be a left RG-module. Describe the dual left RG-module M^* above.

(h) Compare the two constructions M^* and the dual $M^\vee$ (recalled in Question 6), especially when G is finite and R is a field.

(i) Describe a natural transformation $\varpi : \mathrm{id} \longrightarrow (-)^{**}$, as functors from A–Mod to itself.

(j) Show that ϖ_P is an isomorphism when P is a projective left A-module.

Question 6. Let R be commutative and G be a group. Let M and N be two left RG-modules.

(a) Show that $M \otimes_R N$ is a RG-module via $g \cdot (m \otimes n) = (gm) \otimes (gn)$.

(b) Show that $\mathrm{Hom}_R(M, N)$ is a RG-module via $(g\alpha)(m) = g \cdot \alpha(g^{-1} \cdot m)$.

(c) Explicit the above when $N = R$ with trivial G-action. This defines the *dual representation* $M^\vee$ of M (a.k.a. the *contragredient* representation).

(d) Show that there is a natural homomorphism $M^\vee \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)$ and find conditions for this to be an isomorphism.

Question 7. Find adjoints on both sides to the forgetful functor $U : RG\text{–Mod} \to R\text{–Mod}$ which forgets the G action and to the functor $T : R\text{–Mod} \to RG\text{–Mod}$ in the other direction which equips any R-module with the trivial action.

Question 8.

(a) Show that there is no group G such that $\mathbb{C}G \simeq \mathrm{M}_2(\mathbb{C})$.

(b) Is there a group G such that $\mathbb{C}G \simeq \mathbb{C} \times \mathrm{M}_2(\mathbb{C})$?

(c) Find equivalent conditions for $\mathbb{C}G$ to be isomorphic to a product of copies of $\mathbb{C}$.

(d) Describe $\mathbb{C}G$ for G of order 6.

Question 9. Let $G = C_3 =< x >$ be cyclic of order 3. Let $\zeta = e^{2\pi/3} \in \mathbb{C}$.

(a) Consider the $\mathbb{R}G$-module $M_1 = \mathbb{C}$ with action $x \cdot m = \zeta \cdot m$. Show that M_1 is isomorphic to $M_2 = \mathbb{C}$ with $x \cdot m = \bar{\zeta} \cdot m$.

(b) Let $N_1 = \mathbb{C}$ and $N_2 = \mathbb{C}$ be the $\mathbb{C}G$-modules defined similarly. Are they isomorphic as $\mathbb{C}G$-modules?

(c) Decompose $\mathbb{R}G$ and $\mathbb{C}G$ as products of simple rings and explain the above.

Question 10. Let A be a ring and M be an artinian and noetherian left A-module. Let $f : M \to M$ be an A-linear endomorphism. Show that there exists a decomposition $M = M_1 \oplus M_2$ such that $f : M_1 \oplus M_2 \longrightarrow M_1 \oplus M_2$ becomes $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ with $f_1 : M_1 \to M_1$ nilpotent and $f_2 : M_2 \to M_2$ an isomorphism.