Math 210C Homework 6

Question 1. Describe the isomorphism classes of simple A-modules, where $A = \mathrm{M}_n(D) \times \mathrm{M}_n(D)$, for $n \geq 1$ and D a division algebra.

Question 2. Let $G = \mathbb{Z}/p$ and k a field of characteristic not p. Exhibit the decomposition predicted by the Artin-Wedderburn theorem and describe all kG-modules.

Question 3. Let p be a prime, G a p-group and $k = \mathbb{F}_p$. Compute $\mathrm{Rad}(kG)$.

Question 4. Let A be ring.

(a) Let B be another ring. What is $\mathrm{Rad}(A \times B)$?

(b) Show that A is semisimple if and only if $\mathrm{Rad}(A) = 0$ and A is artinian. [Hint: Get an injection of A into a direct sum $\oplus_i (A/\mathfrak{m}_i)$ of simple modules.]

(c) Let $\bar{A} = A/\mathrm{Rad}(A)$. What is $\mathrm{Rad}(\bar{A})$?

(d) Let A be artinian. Show that $A/\mathrm{Rad}(A)$ is semi-simple.

(e) Let A be artinian. Show that $\mathrm{Rad}(A)$ is nilpotent: $\exists\, n \geq 1$ such that $\mathrm{Rad}(A)^n = 0$. [Hint: For n big enough, $\mathrm{Rad}(A)^n = \mathrm{Rad}(A)^{n+1} = ... =: I$. Then, if *ab absurdo* $I \neq 0$, apply a Nakayamian argument to a minimal left ideal J such that $IJ \neq 0$.]

(f) Suppose that A is a commutative, finitely generated k-algebra, for k a field. Show that $\mathrm{Rad}(A) = \sqrt{0}$. [Hint: If $s \in A$ is not nilpotent then $A[\frac{1}{s}]$ is also finitely generated and the maps $\mathrm{Spec}(f)$, for f a homomorphism of such algebras, enjoy a special property.]

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Question 5. Let A be a ring.

(a) Find a characterization of A being artinian simple in terms of the category A–Mod.

(b) Using Morita, deduce that if A is artinian simple then so is $\mathrm{M}_n(A)$.

Question 6. Show that the Weyl algebra is simple but not semi-simple.

Question 7. Let A be a ring.

(a) Show that $\mathrm{Rad}(A) = \{x \in A | \forall\, a \in A,\ 1 + ax \in A^\times\} = \{x \in A | \forall\, b \in A,\ 1 + xb \in A^\times\}$.

(b) A two-sided ideal I of A is said to be *radical* if for every $x \in I$, $1 + x \in A^\times$. Show that $\mathrm{Rad}(A)$ is the largest radical two-sided ideal of A.

(c) Show that every *nil ideal* I of A (a nil ideal is an ideal in which each element is nilpotent) is a radical two-sided ideal of A; hence every such I is contained in $\mathrm{Rad}(A)$.

Question 8 (**The Grothendieck Group**). Let A be a ring, and let $\mathbf{P}(A)$ be the set of isomorphism classes of finitely generated projective A-modules. (Why is $\mathbf{P}(A)$ a set?)

(a) Show that direct sum of modules (with identity 0) makes $\mathbf{P}(A)$ into an abelian monoid.

(b) Let M be an abelian monoid. The *group completion* of M, denoted $M^{-1}M$, is an abelian group along with a monoid morphism $\varphi : M \longrightarrow M^{-1}M$ such that for any monoid morphism $\alpha : M \longrightarrow G$, where G is an abelian group, there is a unique homomorphism $\widetilde{\alpha} : M^{-1}M \longrightarrow G$ such that $\widetilde{\alpha} \circ \varphi = \alpha$. Prove that the group completion is unique (up to isomorphism), and that if M is an abelian group, then $M \cong M^{-1}M$.

(c) Show that $M^{-1}M$ can be realized as $M \times M / \sim$ where $(x, y) \sim (x', y')$ if there exists $z \in M$ such that $x + y' + z = x' + y + z$, with the obvious addition. (Think of the class $[x, y] \in M^{-1}M$ of $(x, y) \in M \times M$ as a formal difference $x - y$.)

(d) Let $(\mathbb{N}, +)$ be the natural numbers (with 0). Find $\mathbb{N}^{-1}\mathbb{N}$.

(e) Let $\varphi : M \longrightarrow N$ be a monoid morphism. Show that φ induces a group homomorphism $\Phi : M^{-1}M \longrightarrow N^{-1}N$, and that Φ is an isomorphism if φ is.

(f) Take a break and eat cookies.

(g) We define the *Grothendieck group* $K_0(A)$ to be the group completion of the monoid $\mathbf{P}(A)$. Show that if $A \longrightarrow B$ is a ring homomorphism, the functor $- \otimes_A B$ induces a group homomorphism $K_0(A) \longrightarrow K_0(B)$.

(h) Show that if A and B are Morita equivalent, then $K_0(A) \cong K_0(B)$ as groups.

(i) Show that if $A = \prod_{i=1}^m A_i$, then $K_0(A) \cong \oplus_{i=1}^m K_0(A_i)$.

(j) Prove that $K_0(\mathrm{M}_n(D)) \cong \mathbb{Z}$, where D is a division ring. Conclude that if A is a semisimple ring, then there is an r with $K_0(A) \cong \mathbb{Z}^r$.

(k) Show that $K_0(\mathbb{Z}) \cong K_0(\mathbb{Q}) \cong \mathbb{Z}$. Show that $\mathbb{Z}$ and $\mathbb{Q}$ are not Morita equivalent