Math 210C Homework 5

Question 1. Let k be a field and $A = k[X]/\langle X^2 \rangle$. Find a non-split exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow k \longrightarrow 0$$

for some A-module M. Conclude that M is not semisimple, even though k and k are.

Question 2 (Morita Equivalence). Let A and B be two rings and P be an A,B-bimodule such that P is finitely generated and projective as left A-module and finitely generated and projective as right B-module and the obvious maps $A \to \mathrm{End}_B(P)$ and $B \to \mathrm{End}_A(P)$ are isomorphisms.

(a) Show that $\mathrm{Hom}_A(P, -)$ and $P \otimes_B -$ induce mutually inverse equivalences between the category A–Mod of left A-modules and the category B–Mod. [Hint: When Q is a finitely generated projective left C-module, show that there is a natural isomorphism $Q^* \otimes_C - \xrightarrow{\sim} \mathrm{Hom}_C(Q, -)$, where $Q^* := \mathrm{Hom}_C(Q, C)$ is a right C-module as usual.]

(b) Show that this equivalence preserves the subcategories of finitely generated modules.

(c) Let R be a ring. Prove that the categories R–Mod and $\mathrm{M}_n(R)$–Mod are equivalent. Appreciate the vast generalization of Question 3 in Homework 4.

Question 3 (Weyl Algebra). Let $k\langle X, Y\rangle$ denote the noncommutative polynomial ring in two variables. Let $W = k\langle X, Y\rangle/\langle YX - XY - 1\rangle$. We call W the *Weyl algebra*. Let $I \subset W$ be a non-zero two-sided ideal.

(a) Show that every element w of W can be written as $w = f_n Y^n + \cdots + f_1 Y + f_0$ where $f_i \in k[X]$ for all i.

(b) Prove that for all i, we have $Y^i X = XY^i + iY^{i-1}$.

(c) Prove that if I contains $w = f_n Y^n + \cdots + f_1 Y + f_0$, with $f_i \in k[X]$, then $f_n \in I$.

(d) Prove that for all j, we have $YX^j = X^j Y + jX^{j-1}$.

(e) Prove that if I contains $f(X) = a_m X^m + \cdots + a_1 X + a_0$, with $a_i \in k$, then $a_m \in I$.

(f) Show that $I = W$. Conclude that W is simple.

(g) Observe that W acts on $k[X]$ by $X \cdot f = Xf$ and $Y \cdot f = \frac{\partial}{\partial X} f$ for any $f \in k[X]$. Show that this defines an injective ring homomorphism $W \hookrightarrow \mathrm{End}_k(k[X])$.

(h) Show that the left ideals $W \cdot Y \supsetneq W \cdot Y^2 \supsetneq W \cdot Y^3 \supsetneq \cdots$ form a strictly descending chain of left ideals. Conclude that W is neither left artinian nor right artinian.

(i) Show that $W \not\cong \mathrm{M}_n(D)$ for any division algebra D and any integer n.

(j) Show that $W \simeq \mathrm{End}_B(W \cdot Y)$ where $B := \mathrm{End}_W(W \cdot Y)$.

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Question 4. Let M be a semi-simple (left) R-module and let S be a simple R-module. Let $D = \mathrm{End}_R(S)$. Recall the natural action of D on $\mathrm{Hom}_R(M, S)$. Show that $\mathrm{Hom}_R(M, S)$ is a finite dimensional D-module (vector space) and show that its dimension is the multiplicity of S in M.

Question 5. Let A and B be two rings. Show that any additive equivalence between A–Mod and B–Mod is given as in Problem 2, for a suitable choice of P. [Hint: $B \mapsto ?$ in A–Mod.]

Question 6. Show that a simple ring R is not necessarily simple as a (left) R-module. Identify the simple rings R which are simple as a (left) R-module.

Question 7.

(a) For which rings k and which groups G is the ring algebra kG simple?

(b) Let D be a division algebra. For which (left) D-modules E is the ring $\mathrm{End}_D(E)$ simple?

Question 8. Let A be a ring and $n \geq 1$.

(a) Determine all two-sided ideals of $M_n(A)$.

(b) Show that if A is simple then so is $M_n(A)$.