Math 210C Homework 4

Question 1. Let E/F be a field extension. We call *transcendence basis* of E over F a subset $\{x_i\}_{i\in I} \subset E$ such that the x_i are algebraically independent over F and such that E is an algebraic extension of the subfield $F(x_i)$ they generate. Prove the following:

(a) A transcendence basis is the same thing as a maximal subset of E (with respect to inclusion) which is algebraically independent over F.

(b) Any field extension E/F has a transcendence basis.

(c) Any two transcendence bases for E/F have the same cardinality. This cardinality is called the *transcendence degree* of E/F, usually denoted $\mathrm{trdeg}_F(E)$ or $\mathrm{trdeg}(E/F)$.

(d) What is the transcendence basis (and degree) if E/F is an algebraic extension?

Question 2. Find the transcendence degree of $\mathrm{Frac}(D)$ (the fraction field of D) over $\mathbb{R}$ in the following cases:

(a) $D = \mathbb{R}[X,Y]/\langle X^2 + Y^2 - 1\rangle$.

(b) $D = \mathbb{R}[X,Y]/\langle Y^2 - X^3\rangle$.

(c) $D = \mathbb{R}[X,Y]/\langle Y^2 - X^2 - X^3\rangle$.

(d) $D = \mathbb{R}[X,Y,Z]/\langle X^2 + Y^2 + Z^2 - 1\rangle$.

Question 3. Describe all finitely generated $M_n(D)$-modules, for D a division algebra, $n \geq 1$.

Question 4.

(a) Show that the quaternion algebra $(\frac{-1,-1}{\mathbb{R}})$ is a division algebra, hence cannot be isomorphic to $\mathrm{M}_2(\mathbb{R})$.

(b) Show that $(\frac{-1,-1}{\mathbb{C}})$ is isomorphic to $\mathrm{M}_2(\mathbb{C})$.

* * *

Question 5. Show by example that the statement of the strong Nullstellensatz is false when the ground field is not algebraically closed.

Question 6. For R a domain, let $\mathrm{Frac}(R)$ denote the fraction field of R.

(a) Show that if $A \subset B$ is an integral extension of domains then $\mathrm{Frac}(A) \subset \mathrm{Frac}(B)$ is algebraic.

(b) Show that if A is a finitely generated k-algebra which is a domain, then $\mathrm{Frac}(A)$ is an algebraic extension of a field of rational functions $k(T_1, ..., T_d)$ for some d.

(c) Express the above integer d in terms of transcendence degree.

Question 7. Let k be an algebraically closed field and $f_1, \dots, f_m \in k[X_1, \dots, X_n]$. Prove that $f_1, \dots, f_m$ have no common zero in k^n if and only if there exists $p_1, \dots, p_m \in k[X_1, \dots, X_n]$ such that $\sum_{i=1}^{m} p_i f_i = 1$. Show that this fails for k not algebraically closed.

Question 8. What can you say about the center of a simple ring?