Math 210C Homework 3

Question 1. Recall the Artin-Rees Lemma from Homework 5, Question 2 in Math 210B.

(a) Prove that if $(A, \mathfrak{m})$ is a local noetherian ring, then

$$\bigcap_{n \geq 0} \mathfrak{m}^n = 0. \tag{1}$$

(b) Prove that if M is a finitely generated A-module with $(A, \mathfrak{m})$ local noetherian, then

$$\bigcap_{n \geq 0} \mathfrak{m}^n M = 0. \tag{2}$$

(c) If A is a noetherian domain and $I \subsetneq A$ a proper ideal, then

$$\bigcap_{n \geq 0} I^n = 0. \tag{3}$$

Question 2. Let k be a field, and i and j relatively prime positive integers. Find a Noether normalization for the ring $A = k[X, Y]/ < X^i - Y^j >$.

Question 3. Recall the p-adic integers $\hat{\mathbb{Z}}_p$.

(a) Prove that $\hat{\mathbb{Z}}_p$ is a local domain with maximal ideal $p \cdot \hat{\mathbb{Z}}_p$.

(b) Prove that $\hat{\mathbb{Z}}_p$ is a DVR and exhibit the valuation.

(c) Let $\mathbb{Q}_p$ be the field of fractions of $\hat{\mathbb{Z}}_p$. Prove that $\mathbb{Q}_p = \hat{\mathbb{Z}}_p[\frac{1}{p}]$ so that every $x \in \mathbb{Q}_p$ is of the form $x = \frac{a}{p^n}$ with $a \in \hat{\mathbb{Z}}_p$.

(d) Prove that for $x \in \mathbb{Q}_p$, x can be written as

$$x = \sum_{i=n}^{\infty} a_i p^i \tag{4}$$

with $a_i \in \{0, \ldots, p-1\}$ and $n \in \mathbb{Z}$.

Question 4. Let $D = \mathbb{R}[X,Y]/ < X^2 + Y^2 - 1 >= \mathbb{R}[x,y]$ and $\mathfrak{m} =< x, y-1 >$.

(a) Show that $\mathfrak{m} \oplus \mathfrak{m} \simeq D^2$. Hint: Contemplate the 2×2 matrix $\begin{pmatrix} x & -y+1 \\ y-1 & x \end{pmatrix}$.

(b) Show that $\mathfrak{m}$ is not principal but that $\mathfrak{m}^2$ is.

(c) Show that $\mathfrak{m} \otimes_D \mathfrak{m} \simeq D$.

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Question 5. Let $A = k[X,Y,Z]/ < XY - Z^2 >= k[x,y,z]$.

(a) Find a Noether normalization of A.

(b) Let $\mathfrak{m} =< x, y, z >\subset A$. Prove that $\mathfrak{m}$ cannot be generated by two elements. (Hint: Consider the vector space $\mathfrak{m}/\mathfrak{m}^2$.)

Question 6. Let k be algebraically closed. Recall that $Z(I) := \{x \in k^n \mid f(x) = 0 \ \forall f \in I\}$.

(a) Show that the sets $Z(I)$ form a closed set topology on the space k^n. We again call this topology the Zariski topology. If $V \subset k^n$ has $V = Z(I)$ for some I, we say V is an *affine variety*, or an *affine algebraic set*.

(b) Justify the name "Zariski topology" by turning the bijection $k^n \to \mathrm{Max}(k[X_1, \ldots, X_n])$ into a homeomorphism for the subspace topology on the latter: $\mathrm{Max} \subset \mathrm{Spec}$.

(c) Let $Y \subset k^n$ be a subset. Define $\mathcal{I}(Y) = \{f \in k[X_1, \ldots, X_n] \mid f(y) = 0 \ \forall y \in Y\}$. Show that $Z(\mathcal{I}(Y)) = \overline{Y}$ and that $\mathcal{I}(Z(I)) = \sqrt{I}$. Conclude that Z and $\mathcal{I}$ give an inclusion-reversing bijection between closed subsets of k^n and radical ideals of $k[X_1, \ldots, X_n]$. (This is another way of stating the Nullstellensatz.)

Question 7. (a) Let $A \subset B$ be an extension of rings. Let $C = \overline{A}^B \subset B$ be the integral closure of A in B. Let $S \subset A$ be a multiplicative subset. Show that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

(b) Show that a domain D is integrally closed if and only if $D_{\mathfrak{p}}$ is integrally closed for every $\mathfrak{p} \in \mathrm{Spec}(D)$ if and only if $D_{\mathfrak{m}}$ is integrally closed for every $\mathfrak{m} \in \mathrm{Max}(D)$.

Question 8. Show that every ideal of a Dedekind domain D is a projective D-module.