Math 210C Homework 2

Question 1. Let k be a field and $n \geq 1$.

(a) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$. Denote evaluation at α by $ev_\alpha : k[X_1, \ldots, X_n] \longrightarrow k$. Prove that its kernel is the ideal $\mathfrak{m}_\alpha := < X_1 - \alpha_1, \ldots, X_n - \alpha_n >$.

(b) Find a bijection between k^n and the subset of $\mathrm{Max}(k[X_1, \ldots, X_n])$ of those maximal ideals $\mathfrak{m}$ whose residue field is isomorphic to k (via the obvious map from k):

$$k \xrightarrow{\sim} k[X_1, \ldots, X_n]/\mathfrak{m}\,.$$

(c) Let $I \subset k[X_1, \ldots, X_n]$ be an ideal and consider the finitely generated k-algebra $A = k[X_1, \ldots, X_n]/I$. Find a bijection between the set $\{\mathfrak{m} \in \mathrm{Max}\, A \mid k \xrightarrow{\sim} A/\mathfrak{m}\}$ and the zero set of the ideal I, namely

$$Z(I) := \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I\}, \tag{1}$$

which coincides with the bijection of (b) when $I = 0$.

Question 2. Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Question 3. Prove that a UFD is integrally closed.

Question 4. Let k be a field, and consider the ring $A = k[X, Y]/ < Y^2 - X^3 >$.

(a) Prove that the ring A is not integrally closed, and is therefore not a UFD.

(b) Prove that $A \cong k[T^2, T^3] \subset k[T]$ and show that $k[T]$ is integral over A.

(c) For the inclusion of rings $k[T^2, T^3] \hookrightarrow k[T]$, prove that the associated map $\mathrm{Spec}(k[T]) \longrightarrow \mathrm{Spec}(k[T^2, T^3])$ is a bijection.

(d) What about $k[T^2] \subset k[T^2, T^3]$?

Question 5. Show that $\mathbb{Z} \subset \mathbb{Z}[i]$ is an integral extension. Discuss the induced map on spectra. [Recall we saw Zagier's one-sentence proof that if p is a prime $p \equiv 1 \bmod 4$, there exists $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = p$.]

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Question 6.

(a) Prove that if A is a domain, then $\operatorname{Spec} A$ has a dense point.

(b) Prove that a point $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if $\mathfrak{p} \subset A$ is a maximal ideal.

(c) Let $Y \subset \operatorname{Spec} A$ be a subset. We define the ideal of Y, denoted $\mathcal{I}(Y)$, by

$$\mathcal{I}(Y) = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}. \tag{2}$$

Prove that $\mathcal{I}(Y) = \mathcal{I}(\overline{Y})$ and that $\mathcal{I}(Y_1) = \mathcal{I}(Y_2)$ if and only if $\overline{Y_1} = \overline{Y_2}$.

(d) Prove that $V(\mathcal{I}(Y)) = \overline{Y}$ and that $\mathcal{I}(V(I)) = \sqrt{I}$.

Question 7. Let $f : A \longrightarrow B$ be a homomorphism of rings.

(a) If f is surjective, prove that the associated morphism $f^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is injective.

(b) If f is injective, prove that the associated morphism $f^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is dominant (i.e. has dense image).

(c) Show by example that f can be injective but f^* need not be surjective.

Question 8. Let M be an A-module. Define the *support* of an element $m \in M$ as $\operatorname{supp}(m) = \{\mathfrak{p} \in \operatorname{Spec} A \,|\, \frac{m}{1} \neq 0 \ in \ M_{\mathfrak{p}}\}$. Define the support of the module as $\operatorname{supp}(M) = \{\mathfrak{p} \in \operatorname{Spec} A | M_{\mathfrak{p}} \neq 0\}$.

(a) Show that $\operatorname{supp}(m) = V(\operatorname{ann}_A(m))$.

(b) Show that $\operatorname{supp}(M) = \cup_{m \in M} V(\operatorname{ann}_A(m))$.

(c) Show that if M is finitely generated, then $\operatorname{supp}(M) = V(\operatorname{ann}_A(M))$ and is therefore a closed subset of $\operatorname{Spec} A$.

(d) Show that the conclusion of (c) is false if M is not finitely generated.

(e) Prove that $M \neq 0$ if and only if $\operatorname{supp}(M) \neq \emptyset$.

Question 9. Let $d \in \mathbb{Z}$. Show that $\mathbb{Z} \subset \mathbb{Z}[\sqrt{d}]$ is an integral extension. Give examples of primes in $\operatorname{Spec}(\mathbb{Z})$ which have multiple preimages in $\operatorname{Spec}(\mathbb{Z}[\sqrt{d}])$, for explicit values of d.

Question 10. Let k be a field and let $k[X,Y]/ < Y^2 - X^2 - X^3 >= k[x,y]$. Prove that the extension $k[X] \cong k[x] \subset k[x,y]$ is integral. What can you say about the induced map on $\operatorname{Spec}(-)$?