Math 210C Homework 1

Question 1. For each subset $E \subset A$, let $V(E) = \{\mathfrak{p} \in \operatorname{Spec}(A) | E \subset \mathfrak{p}\}$. Prove the following:

(a) If I is the ideal generated by E, then $V(E) = V(I) = V(\sqrt{I})$.

(b) $V(\{0\}) = \operatorname{Spec}(A)$ and $V(\{1\}) = \emptyset$.

(c) If $\{E_i\}_{i\in\mathcal{A}}$ is a family of subsets of A, then $V\left(\bigcup_{i\in\mathcal{A}} E_i\right) = \bigcap_{i\in\mathcal{A}} V(E_i)$.

(d) If $\{I_i\}_{i\in\mathcal{A}}$ is a family of ideals of A, then $V\left(\sum_{i\in\mathcal{A}} I_i\right) = \bigcap_{i\in\mathcal{A}} V(I_i)$.

(e) $V(I \cdot J) = V(I \cap J) = V(I) \cup V(J)$ for any ideals I and J of A.

(f) The $V(I)$ form the closed subsets of a topology on $\operatorname{Spec}(A)$, called the *Zariski topology*.

(g) Let $f : A \longrightarrow B$ be a homomorphism of commutative rings, and $\mathfrak{q}$ a prime ideal in B. Then $f^{-1}(\mathfrak{q})$ is a prime ideal in A.

(h) $\operatorname{Spec}(-)$ defines a contravariant functor from commutative rings to topological spaces.

(i) If $\mathfrak{m}$ is a maximal ideal in B, then $f^{-1}(\mathfrak{m})$ need not be maximal.

Question 2. For each $s \in A$, denote $D(s) = \{\mathfrak{p} \in \operatorname{Spec}(A) \,|\, s \notin \mathfrak{p}\}$. We call these sets *principal open sets*. Show that the principal open sets form a basis of open sets for the Zariski topology, and that the following facts are true:

(a) $D(s) \cap D(t) = D(st)$.

(b) $D(s) = \emptyset$ if and only if s is nilpotent.

(c) $D(s) = \operatorname{Spec}(A)$ if and only if s is a unit.

(d) $D(s) \subset D(t)$ if and only if $\exists\, n \geq 1$ such that $t \,|\, s^n$.

(e) $D(s) = D(t)$ if and only if $\sqrt{As} = \sqrt{At}$.

Question 3.

(a) Describe the topological space $\operatorname{Spec}(\mathbb{Z})$.

(b) Describe the topological space $\operatorname{Spec}(\mathbb{Z}[T])$.

Question 4. Let M be a finitely generated R-module and $I \subset R$ be an ideal (see Question 7 for the definition of $\mathrm{Rad}(R)$).

(a) Show that $M \otimes_R R/I = 0$ and $I \subset \mathrm{Rad}(R)$ implies $M = 0$.

(b) If $N \subset M$ is a submodule and $M = IM + N$ where $I \subset \mathrm{Rad}(R)$, then $M = N$.

Assume moreover that R is local with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

(c) Show that $k \otimes_R M = 0$ implies $M = 0$.

(d) Let $x_1, \ldots, x_m \in M$ be elements whose classes $\bar{x}_1, \ldots, \bar{x}_m \in M/\mathfrak{m}M$ form a basis for the k-vector space $M/\mathfrak{m}M$. Show that $x_1, \ldots, x_m$ generate M as an R-module.

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Question 5.

(a) Suppose R is a ring such that for every $x \in R$, there is an integer $n = n(x) > 1$ such that $x^n = x$. Prove that every prime ideal of R is maximal.

(b) Let R be a ring and $\mathfrak{p}$ a prime ideal of R. Suppose $I_1, \ldots, I_n$ are ideals such that $\mathfrak{p} \supset I_1 \cdot I_2 \cdots I_n$. Show there exists a k such that $\mathfrak{p} \supset I_k$.

(c) Show that in an Artinian ring (e.g. a finite ring or a finite dimensional algebra over a field), there are only finitely many prime ideals and that they are all maximal.

Question 6. Let R be an integral domain and F its field of fractions. Prove that:

$$R = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} R_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}} \tag{1}$$

as subsets of F. [Hint: for $a \in \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}}$, consider the ideal $I_a = \{d \in R \,|\, da \in R\}$.]

Question 7. The *radical* of an ideal $I \subset A$ is defined by $\sqrt{I} = \{x \in A \,|\exists\, n \text{ with } x^n \in I\}$. The *Jacobson radical* of A is defined by $\mathrm{Rad}(A) = \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} \mathfrak{m}$.

(a) Prove that $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$.

(b) Show that the nilradical $\sqrt{0}$ is contained in $\mathrm{Rad}(A)$ but that equality is far from true.

(c) Prove that $x \in \mathrm{Rad}(R)$ if and only if $1 - xy$ is a unit for all $y \in A$.

Question 8. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m \in \mathrm{Spec}(R)$. Suppose I is an ideal such that

$$I \subset \bigcup_{i=1}^{m} \mathfrak{p}_i. \tag{2}$$

Prove that $I \subset \mathfrak{p}_k$ for some k.