MATH 210B: HOMEWORK 9

Problem 82. What is the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$?

Problem 83. Let F be a field, and let $A = F[X_1, \ldots, X_n]/(X_1^2 + \cdots + X_n^2)$. Let K be the field of fractions of A. What is the transcendence degree of K over F?

Problem 84. Let F be a field, and let $A = F[X, Y, Z]/(X + Y + Z, XYZ - 1)$. Let K be the field of fractions of A. What is the transcendence degree of K over F?

Problem 85. For any commutative ring A, prove that the Zariski topology on $\operatorname{Spec} A$ is actually a topology. That is, prove that $\varnothing$ and $\operatorname{Spec} A$ are open, and that open sets are closed under arbitrary union and finite intersection.

Problem 86.

(a) Prove that construction of the spectrum of a commutative ring gives rise to a functor $\operatorname{Spec} : \mathbf{CommRing}^{\mathrm{op}} \to \mathbf{Top}$.
(b) Prove that Spec is neither full nor faithful.

Problem 87. Let A be a commutative ring. Prove that $\operatorname{Spec} A$ is not connected if and only if $A \cong A_1 \times A_2$ for some nonzero rings A_1, A_2.

Problem 88.

(a) Let A be an integral domain. Prove that $\operatorname{Spec} A$ is irreducible, that is, it cannot be written as a union of two proper closed subsets.
(b) Give an example of a ring R such that $\operatorname{Spec} R$ not irreducible.

Problem 89.

(a) Describe $\operatorname{Spec} k[X]$, where k is a field.
(b) Describe $\operatorname{Spec} k[X]/(X^2)$ and $\operatorname{Spec} k[X]_{(X)}$.
(c) Repeat (a) and (b) for k an algebraically closed field.

Problem 90. Describe $\operatorname{Spec} \mathbb{Z}$ and $\operatorname{Spec} \mathbb{Z}[X]$.

Problem 91. Describe $\operatorname{Spec} \mathbb{R}[X]$ and $\operatorname{Spec} \mathbb{R}[X]/(X^2 + 1)$.