MATH 210B: HOMEWORK 8

Problem 72. Let $\operatorname{char} F = p$ and K/F be a finite purely inseparable extension. Compute formulas for the norm $\mathrm{N}_{K/F} : K^\times \to F^\times$ and the trace $\mathrm{Tr}_{K/F} : K \to F$.

Problem 73. Given two finite extensions $F \subset K \subset L$, show that $\mathrm{N}_{L/F} = \mathrm{N}_{K/F} \circ \mathrm{N}_{L/K}$ and $\mathrm{Tr}_{L/F} = \mathrm{Tr}_{K/F} \circ \mathrm{Tr}_{L/K}$. Hint: the separable case was done in class.

Problem 74. Let K/F be a finite field extension.

(a) Show that $K \times K \to F$, $(x, y) \mapsto \mathrm{Tr}_{K/F}(xy)$ defines a F-bilinear pairing.

(b) Show that the above pairing is nondegenerate if and only if K/F is separable.

Problem 75. Let $K = \mathbb{Q}(\theta)$, where θ is a root of $X^3 - 2X + 6$. Compute $\mathrm{N}_{K/\mathbb{Q}}(\theta)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\theta)$.

Problem 76. Let $d \equiv 1 \mod 4$ be a squarefree integer. Let $L = \mathbb{Q}(\sqrt{d})$ and let $\alpha = \frac{1+\sqrt{d}}{2} \in L$. Prove that $\mathrm{N}_{L/\mathbb{Q}}(\alpha) = (1-d)/4$ and $\mathrm{Tr}_{L/\mathbb{Q}}(\alpha) = 1$.

Problem 77. Let $E/\mathbb{Q}$ be the splitting field of $f(X) = X^6 - 1$. Let $\zeta \in E$ be a primitive element, so $E = \mathbb{Q}(\zeta)$. Compute $\mathrm{N}_{E/\mathbb{Q}}(\zeta)$ and $\mathrm{Tr}_{E/\mathbb{Q}}(\zeta)$.

Problem 78. Let F be a field and K/F a finite cyclic extension of degree n with Galois group $G = \langle \sigma : \sigma^n = 1 \rangle$.

(a) Prove that there exists $z \in K$ such that $\mathrm{Tr}_{K/F}(z) \neq 0$.

(b) Let $x \in K$ satisfy $\mathrm{Tr}_{K/F}(x) = 0$. For a z as above, let

$$y = \frac{1}{\mathrm{Tr}_{K/F}(z)} \Big[x \cdot z^{\sigma} + (x + \sigma(z)) \cdot z^{\sigma^2} + \cdots + (x + \sigma(x) + \cdots + \sigma^{n-2}(x)) \cdot z^{\sigma^{n-1}} \Big].$$

Prove that $x = y - \sigma(y)$.

(c) Conclude the additive form of Hilbert's Theorem 90 from the above: for $x \in K$, $\mathrm{Tr}_{K/F}(x) = 0$ if and only if there exists $y \in K$ such that $x = y - \sigma(y)$.

Problem 79. Let F be a field of characteristic p and K/F a finite cyclic extension of degree p^{m-1} with Galois group $G = \langle \sigma : \sigma^{p^{m-1}} = 1 \rangle$, where $m \geq 2$.

(a) Let $\beta \in K$ such that $\mathrm{Tr}_{K/F}(\beta) = 1$. Show that there exists $\alpha \in K$ such that $\sigma(\alpha) - \alpha = \beta^p - \beta$.

(b) Prove that $X^p - X - \alpha$ is irreducible in $K[X]$.

(c) If θ is a root of this polynomial, prove $F(\theta)$ is a Galois, cyclic extension of degree p^m over F, and that its Galois group is generated by τ such that $\tau|_K = \sigma$ and $\tau(\theta) = \theta + \beta$.

Problem 80.

(a) Let K/F be a finite extension of finite fields. Prove that the norm and trace maps are surjective.

(b) Prove that the norm map $\mathrm{N}_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \to \mathbb{R}^\times$ is not surjective.

Problem 81. Calculate the Galois group of the following polynomials over $\mathbb{Q}$, then determine which are solvable by radicals.

(a) $f(X) = X^5 + X + 1$
(b) $g(X) = X^5 - X + 1$
(c) $h(X) = X^5 + 20X + 32$
(d) $\varphi(X) = X^6 - 2X^3 - 1$
(e) $\psi(X) = X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$
(f) $\rho(X) = X^6 + X^4 + X + 3$