MATH 210B: HOMEWORK 7

Problem 66. Let F be a field of characteristic zero containing the p-th roots of unity for p a prime. Show that the cyclic extensions of degree p of F in any algebraic closure $\overline{F}$ of F are in one-to-one correspondence with the subgroups of order p of $F^\times/(F^\times)^p$.

Problem 67.

(a) Let $K = \mathbb{Q}(\sqrt{-N})$ where $N \in \mathbb{Z}$ and $N > 0$. Show that K cannot be embedded in a cyclic extension of $\mathbb{Q}$ whose degree is divisible by 4.

(b) Let $f(X) = X^4 + 4X^2 + 2$. Prove that f is irreducible over $\mathbb{Q}$ and that the Galois group of f is cyclic.

(c) Let $g(X) = X^4 + 30X^2 + 45$. Let α be a root of g. Prove that $\mathbb{Q}(\alpha)$ is cyclic of degree 4 over $\mathbb{Q}$.

Problem 68. Let n be a positive odd integer and let F be a field of characteristic $p \nmid 2n$. Prove that F contains a primitive n-th root of unity if and only if it contains a primitive $2n$-th root of unity.

Problem 69. Let $\mathbb{F}$ be a finite field, and let $\overline{\mathbb{F}}$ be a fixed algebraic closure. Let K be the subfield of $\overline{\mathbb{F}}$ generated by all roots of unity. Prove that $K = \overline{\mathbb{F}}$.

Problem 70. Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$.

(a) Prove there exists a maximal subfield E of $\overline{\mathbb{Q}}$ not containing $\sqrt{2}$.

(b) Prove that every finite extension of E is cyclic. Your proof should work for any algebraic irrational number $\sqrt[n]{a}$, not just $\sqrt{2}$.

Problem 71. Let F be a field, and let $\overline{F}$ be a fixed algebraic closure. Let σ be an automorphism of $\overline{F}$ that fixes F, and let K be the fixed field of σ. Show that every finite extension of K is cyclic.