MATH 210B: HOMEWORK 5

Problem 43. Let L be the splitting field of $T^4 - 7$ over $\mathbb{Q}$. Show that this extension is simple. Find $x \in L$ such that $L = \mathbb{Q}(x)$.

Problem 44. Give an example of an infinite group G acting on a field E (by field automorphisms) with fixed field $K = E^G$ such that E/K is not an algebraic extension.

Problem 45. Give an example of an infinite separable field extension L/K and a subgroup $H < \mathrm{Aut}(L/K)$ such that $\mathrm{Aut}(L/L^H) \neq H$.

Problem 46. Prove that the polynomial T^4+1 is reducible in $\mathbb{F}_p[T]$ for any prime p.

Problem 47. Prove that every finite field admits a quadratic extension by proving that $\mathbb{F}_q[T]$ contains an irreducible quadratic polynomial for any prime power q.

Problem 48. Let G be a finite group. Prove that there exists a normal, separable field extension K/F in any characteristic such that $\mathrm{Gal}(K/F) = G$.

Problem 49. Let Γ be the Galois group of the polynomial $X^5 - 9X + 3$ over $\mathbb{Q}$. Determine Γ.

Problem 50. Let $p(X)$ be an irreducible cubic polynomial over $\mathbb{Z}$. Prove that the Galois group of $p(X)$ must be S_3 or A_3. Give necessary and sufficient conditions for the Galois group being either.

Problem 51. Pick a non-zero rational number a. Determine all possibilities for the Galois group G of the normal closure of $\mathbb{Q}(\sqrt[4]{a})$ over $\mathbb{Q}$, where $\sqrt[4]{a}$ is the root of $X^4 - a$ of maximal degree over $\mathbb{Q}$.

Problem 52. Assume that L is a Galois extension of the field of rational numbers $\mathbb{Q}$ and that $K \subset L$ is the subfield generated by all roots of unity in L. Suppose that $L = \mathbb{Q}(a)$, where $a^n \in \mathbb{Q}$ for some positive integer n. Show that the Galois group $\mathrm{Gal}(L/K)$ is cyclic.

Problem 53. Let $K \subset L$ be an algebraic extension of fields. An element $a \in L$ is called *abelian* if $K(a)$ is a Galois extension of K with abelian Galois group $\mathrm{Gal}(K(a)/K)$. Show that the set of abelian elements of L is a subfield of L containing K.

Problem 54. Let $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ be an extension of finite fields. Show that the Frobenius automorphism $x \mapsto x^q$ generates the Galois group of this extension. Describe the group, its subgroups and the corresponding sub-fields under the Galois correspondence.