MATH 210B: HOMEWORK 4

Problem 31. Let K_1 and K_2 be two fields normal over F, and assume that both K_1 and K_2 lie in some larger field extension L over F.

(a) Prove that the composite field K_1K_2 is normal over F.
(b) Prove that $K_1 \cap K_2$ is a normal over F.

Problem 32. Determine the splitting field for $X^4 + X^2 + 1$ over $\mathbb{Q}$. What is its degree?

Problem 33. Determine the splitting field for $X^8 - 1$ over $\mathbb{Q}$. What is its degree?

Problem 34. Determine the splitting field for $X^8 - 1$ over $\mathbb{F}_5$. What is its degree?

Problem 35.

(a) Prove that $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ is a normal, separable extension over $\mathbb{Q}$.
(b) Let $K = \mathbb{Q}(\sqrt[3]{2} + \sqrt[3]{3})$. Prove that $K/\mathbb{Q}$ is not normal, then compute the normal closure L of K and find the degree of L over $\mathbb{Q}$.

Problem 36. Let L/F be a field extension which is separable but not normal. Prove that there exists a maximal subextension $L \supset K \supset F$ such that K/F is normal. Interpret this result in the language of groups by using Galois theory.

Problem 37. Let G be a group and $H < G$ a subgroup. Show that there is a maximal subgroup $K < H$ such that $K \triangleleft G$ is normal.

Problem 38. Let L/K be any extension. Prove that the following are equivalent.

(a) Every $x \in L \setminus K$ has a non-separable minimal polynomial.
(b) $\operatorname{char} K = p > 0$ and every $x \in L$ has some iterated pth power in K, that is, $x^{p^n} \in K$ for $n >> 0$. Such $x \in L$ are called *purely inseparable* over K.
(c) $\operatorname{char} K = p > 0$ and every $x \in L$ has a minimal polynomial of the form $T^{p^n} - a$ for some $a \in K$.
(d) $[L : K]_{\text{sep}} = 1$.
(e) L is generated by purely inseparable elements.

In this case, L/K is called *purely inseparable*.

Problem 39. Let L/F be a finite extension.

(a) Prove that there exists a maximal separable subextension K/F contained in L.

(b) Prove that L/K is purely inseparable.
(c) Prove that the set of purely inseparable elements in L over F form a subfield.

Problem 40. Let p be prime, and consider $F = \mathbb{F}_p(X)$ the field of rational functions. Let L be the splitting field of $T^{pq} - X$ for q coprime to p.

(a) Prove that $T^{pq} - X$ is irreducible in $\mathbb{F}_p(X)$.
(b) Calculate K the maximal separable subextension, and prove that L/K is purely inseparable (explicitly).

Problem 41. Recall that a field is called *perfect* if every extension of that field is separable.

(a) Prove that fields of characteristic 0 are perfect.
(b) Let F be a field of characteristic $p > 0$. Prove that the map $\varphi : F \to F$, $\varphi(x) = x^p$, is a field endomorphism.
(c) Prove that F is perfect if and only if φ is an automorphism.

Problem 42. Prove that $\mathbb{F}_{49}$ is perfect by using the Frobenius automorphism.