MATH 210B: HOMEWORK 3

Problem 20. Let k be a field, and let V be a k-vector space. Recall that the dual of V is defined to be $V^* := \operatorname{Hom}_k(V, k)$.

(a) Prove that there is a canonical morphism $V \to V^{**}$.
(b) Prove that this canonical morphism is an isomorphism if and only if V is finite dimensional.

Problem 21. Suppose that V and W are two k-vector spaces, and let $B : V \times W \to k$ be a k-bilinear map.

(a) Using hom-tensor duality, prove that B induces two k-linear maps $L : V \to W^*$ and $R : W \to V^*$.
(b) B is called a *perfect pairing* if both L and R are isomorphisms. Prove that, if V and W are finite dimensional, B is a perfect pairing if and only if it is non-degenerate.
(c) Is every perfect pairing non-degenerate?

Problem 22. Generalize the above in the case of modules over a ring R. Prove that a non-degenerate pairing of finitely generated R-modules need not be perfect.

Problem 23. Let $V = C^\infty(\mathbb{R})$ be the $\mathbb{R}$-vector space of smooth functions $\mathbb{R} \to \mathbb{R}$. For each closed interval $[a, b] \subset \mathbb{R}$, define

$$B(f, [a, b]) = \int_a^b f(x)\, dx.$$

What is the vector space W in this case? Is this a pairing? Is it perfect?

Problem 24. Let $k(X)$ denote the field of rational functions over a field k in the variable X

(a) Prove that $k(X)$ is algebraic over $k(X^n)$ via the canonical embedding. What is its degree?
(b) Prove that $k(X)$ and $k(X^n)$ are (abstractly) isomorphic.

Problem 25. How many distinct subfields of $\mathbb{C}$ are isomorphic to $\mathbb{Q}[X]/(X^4 - 7)$? What is their intersection?

Problem 26. Consider a set of points $P \subset \mathbb{R}^2$. We define the *point field* K of P to be the smallest subfield of $\mathbb{R}$ containing each coordinate of every point in P.

(a) Prove that if the point (x, y) is constructible from P, then x and y are roots of quadratic polynomials over the point field K.
(b) Conclude that given any finite series of constructions $P_0 \subset P_1 \subset \cdots \subset P_n$, the corresponding field extension K_n/K_0 has degree a power of 2.

Problem 27. Prove that it is impossible to construct a square of the same area as a given circle in $\mathbb{R}^2$ using only straightedge and compass.

Problem 28. Prove that it is impossible to construct the edge of a cube that has twice the volume of a given cube using only straightedge and compass.

Problem 29. Prove that it is impossible to trisect an angle using only straightedge and compass.

Problem 30. Prove that the regular 7-gon is not constructible using only straightedge and compass.