Problem 20. Let $k$ be a field, and let $V$ be a $k$-vector space. Recall that the dual of $V$ is defined to be $V^* := \text{Hom}_k(V,k)$.

(a) Prove that there is a canonical morphism $V \to V^{**}$.
(b) Prove that this canonical morphism is an isomorphism if and only if $V$ is finite dimensional.

Problem 21. Suppose that $V$ and $W$ are two $k$-vector spaces, and let $B : V \times W \to k$ be a $k$-bilinear map.

(a) Using hom-tensor duality, prove that $B$ induces two $k$-linear maps $L : V \to W^*$ and $R : W \to V^*$.
(b) $B$ is called a perfect pairing if both $L$ and $R$ are isomorphisms. Prove that, if $V$ and $W$ are finite dimensional, $B$ is a perfect pairing if and only if it is non-degenerate.
(c) Is every perfect pairing non-degenerate?

Problem 22. Generalize the above in the case of modules over a ring $R$. Prove that a non-degenerate pairing of finitely generated $R$-modules need not be perfect.

Problem 23. Let $V = C^\infty(\mathbb{R})$ be the $\mathbb{R}$-vector space of smooth functions $\mathbb{R} \to \mathbb{R}$. For each closed interval $[a, b] \subset \mathbb{R}$, define

$$ B(f, [a, b]) = \int_a^b f(x) \, dx. $$

What is the vector space $W$ in this case? Is this a pairing? Is it perfect?

Problem 24. Let $k(X)$ denote the field of rational functions over a field $k$ in the variable $X$.

(a) Prove that $k(X)$ is algebraic over $k(X^n)$ via the canonical embedding. What is its degree?
(b) Prove that $k(X)$ and $k(X^n)$ are (abstractly) isomorphic.

Problem 25. How many distinct subfields of $\mathbb{C}$ are isomorphic to $\mathbb{Q}[X]/(X^4 - 7)$? What is their intersection?

Problem 26. Consider a set of points $P \subset \mathbb{R}^2$. We define the point field $K$ of $P$ to be the smallest subfield of $\mathbb{R}$ containing each coordinate of every point in $P$. 

(a) Prove that if the point \((x, y)\) is constructible from \(P\), then \(x\) and \(y\) are roots of quadratic polynomials over the point field \(K\).

(b) Conclude that given any finite series of constructions \(P_0 \subset P_1 \subset \cdots \subset P_n\), the corresponding field extension \(K_n/K_0\) has degree a power of 2.

**Problem 27.** Prove that it is impossible to construct a square of the same area as a given circle in \(\mathbb{R}^2\) using only straightedge and compass.

**Problem 28.** Prove that it is impossible to construct the edge of a cube that has twice the volume of a given cube using only straightedge and compass.

**Problem 29.** Prove that it is impossible to trisect an angle using only straightedge and compass.

**Problem 30.** Prove that the regular 7-gon is not constructible using only straightedge and compass.