MATH 210B: HOMEWORK 2

Problem 13. Recall the Weyl algebra over a field k is defined by the quotient $A_2 := k\langle X, Y\rangle$ of the non-commutative polynomial algebra by the (two-sided) ideal $(XY - YX - 1)$. Prove that A_2 is isomorphic to $k[X, Y]$ as a k-vector space. Furthermore, prove that multiplication in $k[X]$ (resp. $k[Y]$) coincides with the multiplication in its image in A_2.

Problem 14. Let A be a $m \times n$ matrix with entries in a PID R. The *Smith normal form* of the matrix is found by computing two invertible square matrices S and T so that $S \cdot A \cdot T$ has nonzero entries $\alpha_1, \ldots, \alpha_r$ only along its diagonal with $\alpha_i \mid \alpha_{i+1}$.To find the Smith normal form of A, one is permitted to exchange rows and columns as well as add multiples of one row to another (these are invertible operations). One may also multiply any row or column by an *invertible* scalar.

(a) Compute the matrix σ_{ij} that exchanges rows r_i and r_j and the matrix $\tau_{k\ell}$ that exchanges columns c_k and c_ℓ. On which side of A do you multiply to perform these operations?
(b) Compute the matrix φ that adds $\alpha \cdot r_i$ to r_j and ρ that adds $\beta \cdot c_k$ to c_ℓ. On which side of A do you multiply to perform these operations?
(c) Using these operations, prove that the Smith normal form exists for any matrix.

Problem 15. Compute S, T, and the Smith normal form of the matrix

$$\begin{pmatrix} 2i & 4+3i & 6 \\ 5 & -2 & 1+4i \end{pmatrix}$$

over the ring $\mathbb{Z}[i]$.

Problem 16. Compute the Smith normal form of the matrix

$$\begin{pmatrix} 2X(X-1)(X-2)^2 & 0 & 0 \\ 0 & 5X-10 & 0 \\ 0 & 0 & 3X-3 \end{pmatrix}$$

with entries in the ring $\mathbb{Q}[X]$. (Hint: recall what it means for $f \mid g$ in this ring.)

Problem 17. Let M be a finitely generated module over a PID R. Then we know that M appears as a quotient the free R-modules

$$0 \to R^n \xrightarrow{f} R^m \to M \to 0.$$

The map f yields an $m \times n$ matrix A with entries in R using the standard basis $\{e_i\}$ for R^n and R^m. Prove that the diagonal entries α_i of the Smith normal form of A give a presentation of M

$$M \cong R/(\alpha_1) \oplus \cdots \oplus R/(\alpha_r),$$

which is isomorphic to the *elementary divisor form* given in the classification theorem:

$$M \cong R^k \oplus R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_\ell^{a_\ell}).$$

Problem 18.

(a) Using the classification theorem, compute the number of non-isomorphic abelian groups of order 168.

(b) Compute the number of non-isomorphic $\mathbb{Z}[i]$-modules of order 168. (Hint: what are the prime elements in this ring?)

Problem 19. Let F be the field with 2 elements and let $R = F[X]$. List up to isomorphism all R-modules with 8 elements that are cyclic (generated by one element).