Problem 0. If you failed to do this last quarter, prove the snake lemma.

Problem 1. A commutative ring $A$ is called local if it has a unique maximal ideal $m$, in which case $k = R/m$ is a field. Prove that a commutative ring $A$ is local if and only if the non-units of $A$ form an ideal. Note: a local ring will always be assumed commutative.

Problem 2. Let $A$ be a commutative ring and $M$ a finitely-generated $A$-module. Let $f : M \to M$ be an $A$-module epimorphism. Prove that $f$ is an isomorphism. Hint: the inverse of $f$ can be obtained a polynomial in $f$.

Problem 3. Let $A$ be a local ring with maximal ideal $m$, and let $M$ and $N$ be finitely-generated $A$-modules. Let $f : M \to N$ be an $A$-module homomorphism. Prove that $f$ is surjective if and only if the map $f_m : M/mM \to N/mN$ is surjective.

Problem 4. Let $A$ be a local ring with maximal ideal $m$, and let $M$ be a finitely-generated $A$-module.
   
   (a) Prove that if $M/mM$ is generated by $x_1, \ldots, x_n$, then so is $M$.
   
   (b) Prove that if $M$ is projective, then it is free.

Problem 5. Let $A$ be a local ring with maximal ideal $m$, and let $M$ and $N$ be finitely-generated $A$-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Problem 6. Let $A = \mathbb{k}[X]$, the polynomial ring of a field $\mathbb{k}$, and consider the $A$-module $M = \mathbb{k}[X, X^{-1}]$. Let $I = \langle X \rangle$.

   (a) Show that $M = IM$, but there is no $f \in R \setminus I$ for which $f \cdot M = 0$.
   
   (b) Why is this consistent with Nakayama’s lemma?

Problem 7. Check that most hypotheses in the above problems are necessary by finding counterexamples. For example, find a counterexample to Problem 2 in the case that $M$ is infinitely generated and find a counterexample to Problem 5 when $A$ is not local.

Problem 8. Prove that if $A$ is a commutative Noetherian local ring with maximal ideal $m$, then

$$\bigcap_{i=1}^{\infty} m^i = \{0\}.$$
Problem 9. Although an artinian ring is necessarily a noetherian ring, find an example of a ring $A$ and artinian module $M$ that is not noetherian.

The following problems rely on the classification of finitely generated modules over a PID.

Problem 10. Show that every finitely-generated $R$-module $M$, for a PID $R$, has a decomposition in the following form:

$$M \cong R^n \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

where $n, m \in \mathbb{N}$, and $a_1 | a_2 | \cdots | a_m$. The number $n$ is called the rank of $M$, and the $a_i$ are called the invariant factors of $M$.

Problem 11. Let $V$ be a finite dimensional vector space over a field $k$, and let $T : V \to V$ be a linear endomorphism. Prove that $V$ has a unique structure as a $k[t]$-module, where $t \cdot v = T(v)$ for any $v \in V$. Prove that the rank of $V$ as a $k[t]$-module is 0.

Problem 12. What are necessary and sufficient conditions on the invariant factors of a linear endomorphism $T$ so that $T$ is diagonalizable over the algebraic closure $k^{\text{alg}}$ of $k$? What about over $k$ itself?