MATH 210B: HOMEWORK 1

Problem 0. If you failed to do this last quarter, prove the snake lemma.

Problem 1. A commutative ring A is called *local* if it has a unique maximal ideal $\mathfrak{m}$, in which case $k = R/\mathfrak{m}$ is a field. Prove that a commutative ring A is local if and only if the non-units of A form an ideal. Note: a local ring will always be assumed commutative.

Problem 2. Let A be a commutative ring and M a finitely-generated A-module. Let $f : M \to M$ be an A-module epimorphism. Prove that f is an isomorphism. Hint: the inverse of f can be obtained a polynomial in f.

Problem 3. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M and N be finitely-generated A-modules. Let $f : M \to N$ be an A-module homomorphism. Prove that f is surjective if and only if the map $f_{\mathfrak{m}} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective.

Problem 4. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely-generated A-module.

(a) Prove that if $M/\mathfrak{m}M$ is generated by $x_1, \ldots, x_n$, then so is M.
(b) Prove that if M is projective, then it is free.

Problem 5. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let M and N be finitely-generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Problem 6. Let $A = k[X]$, the polynomial ring of a field k, and consider the A-module $M = k[X, X^{-1}]$. Let $I = \langle X \rangle$.

(a) Show that $M = IM$, but there is no $f \in R \setminus I$ for which $f \cdot M = 0$.
(b) Why is this consistent with Nakayama's lemma?

Problem 7. Check that most hypotheses in the above problems are necessary by finding counterexamples. For example, find a counterexample to Problem 2 in the case that M is infinitely generated and find a counterexample to Problem 5 when A is not local.

Problem 8. Prove that if A is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$, then

$$\bigcap_{i=1}^{\infty} \mathfrak{m}^i = \{0\}.$$

Problem 9. Although an artinian ring is necessarily a noetherian ring, find an example of a ring A and artinian module M that is not noetherian.

The following problems rely on the classification of finitely generated modules over a PID.

Problem 10. Show that every finitely-generated R-module M, for a PID R, has a decomposition in the following form:

$$M \cong R^n \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

where $n, m \in \mathbb{N}$, and $a_1 \mid a_2 \mid \cdots \mid a_m$. The number n is called the *rank* of M, and the a_i are called the *invariant factors* of M

Problem 11. Let V be a finite dimensional vector space over a field k, and let $T : V \to V$ be a linear endomorphism. Prove that V has a unique structure as a $k[t]$-module, where $t \cdot v = T(v)$ for any $v \in V$. Prove that the rank of V as a $k[t]$-module is 0.

Problem 12. What are necessary and sufficient conditions on the invariant factors of a linear endomorphism T so that T is diagonalizable over the algebraic closure k^{alg} of k? What about over k itself?