# Math 210B Homework 9

**Question 1.** Prove that $\mathbb{F}_p(X,Y)/\mathbb{F}_p(X^p,Y^p)$ has infinitely many subfields $K$ such that $\mathbb{F}_p(X^p,Y^p) \subset K \subset \mathbb{F}_p(X,Y)$. Thus, $\mathbb{F}_p(X,Y)/\mathbb{F}_p(X^p,Y^p)$ is not a simple extension.

**Question 2.** Let $K$ be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(X) = X^6 + 3$.

(a) Show that $K$ contains a primitive $6^{th}$ root of unity.

(b) Show that $K$ is Galois over $\mathbb{Q}$. Find $\mathrm{Gal}(K/\mathbb{Q})$.

(c) Determine the number of fields $F$ of degree 3 over $\mathbb{Q}$ with $F \subset K$.

**Question 3.**

(a) Find the Galois group of the polynomial $X^3 - X^2 - 4$.

(b) Find the Galois group of the polynomial $X^4 - 25$.

**Question 4.** Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$, $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$, and $F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $\mathrm{Gal}(K/F_1) \cong \mathbb{Z}/8\mathbb{Z}$, $\mathrm{Gal}(K/F_2) \cong D_8$, and $\mathrm{Gal}(K/F_3) \cong Q_8$.

**Question 5 (Inverse Galois Theory).**

(a) Let $G$ be any finite group. Prove there exists a Galois extension $L/K$ such that $\mathrm{Gal}(L/K) \cong G$.

(b) Let $A$ be a finite abelian group. Prove that there exists a Galois extension $K/\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong A$. (Hint: You may freely use the fact that there are infinitely many primes $p \equiv 1 \mod n$ for any $n > 1$).

(c) Let $A$ be a finite abelian group, and let $k$ be some fixed number field of finite degree over $\mathbb{Q}$. Prove that there exists a Galois extension $K/k$ such that $\mathrm{Gal}(K/k) \cong A$.

**Remark 0.1.** The problem described in parts (b) and (c) is known as the Inverse Galois Problem. We in fact know the result to be true for solvable groups and for most simple groups, but for arbitrary groups the problem remains unsolved.

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**Question 6.** Suppose $K$ is a Galois extension of $F$ of degree $p^n$ for some prime $p$ and some $n \geq 1$. Show that there are Galois extensions of $F$ contained in $K$ of degrees $p$ and $p^{n-1}$.

**Question 7.** Determine the Galois group of the normal closure of $\mathbb{Q}(\sqrt{-3}+\sqrt[3]{2})/\mathbb{Q}$.

**Question 8.**

(a) Determine the degree $[\mathbb{R}:\mathbb{Q}]$.

(b) Determine the group $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{R})$.

**Question 9.** Prove that $\cos(\frac{2\pi}{31})$ is algebraic over $\mathbb{Q}$ and find its degree.

**Question 10.** Consider the field $K=\mathbb{Q}(\zeta,\alpha,\beta)$, where $\zeta\neq 1$ is a cube root of 1, $\alpha=\sqrt[3]{2}$, and $\beta=\sqrt[3]{5}$

(a) Show that $K/\mathbb{Q}$ is a Galois extension and compute $[K:\mathbb{Q}]$.

(b) Show that there are automorphisms $\rho$, $\sigma$, and $\tau$ of $K/\mathbb{Q}$ such that

$$\begin{aligned}\rho(\zeta)=\zeta^{-1},\rho(\alpha)=\alpha,\rho(\beta)=\beta\\ \sigma(\zeta)=\zeta,\sigma(\alpha)=\zeta\alpha,\sigma(\beta)=\beta\\ \tau(\zeta)=\zeta,\tau(\alpha)=\alpha,\tau(\beta)=\zeta\beta.\end{aligned}\tag{1}$$

(c) Find the orders of each of these automorphisms.

(d) Show that $\sigma\tau=\tau\sigma$, $\rho\sigma\rho^{-1}=\sigma^2$, and that $\rho\tau\rho^{-1}=\tau^2$. Use this to determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(e) How many subfields $\mathbb{Q}\subset L\subset K$ are there with $[L:\mathbb{Q}]=9$? How many subfields $\mathbb{Q}\subset M\subset K$ are there with $[M:\mathbb{Q}]=6$?