Math 210B Homework 7

Question 1. Let E/F and K/F be two finite field extensions of F contained in a larger extension L, and consider the composite field $M = EK$ over F. Show that if E/F is normal then so is M/K.

Question 2.

(a) Let $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Prove that $K/\mathbb{Q}$ is a normal, separable extension.

(b) Let $K = \mathbb{Q}(\sqrt[4]{2})$. Prove that $K/\mathbb{Q}$ is not a normal extension. Let M be the normal closure of K. Find M and $[M : \mathbb{Q}]$.

Question 3. Let K/F be a separable quadratic extension.

(a) Prove that if $\mathrm{char}(F) \neq 2$, then $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^2 \in F$.

(b) Prove that if $\mathrm{char}(F) = 2$, then $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^2 + \alpha \in F$.

Question 4. Let ζ_n be a primitive n^{th} root of unity. Recall that $X^n - 1 = \Pi_{k=1}^{n}(X - \zeta_n^k)$.

(a) Prove that $\zeta_n^{n/d}$ is a primitive d^{th} root of unity.

(b) Use the above equation to show that $X^n - 1 = \prod_{d|n} \Phi_d(X)$, where

$$\Phi_d(X) = \prod_{1 \leq j \leq d,\ \gcd(j,d)=1} (X - \zeta_n^{j\cdot(n/d)}). \tag{1}$$

Use this to prove the number theory identity

$$n = \sum_{d|n} \varphi(d). \tag{2}$$

Here $\varphi(m) = \#\{\, k \mid 1 \leq k \leq m, \gcd(k,m) = 1\} = \left|(\mathbb{Z}/m\mathbb{Z})^{\times}\right|$ is the Euler function.

(c) Use part (b) to find $\Phi_6(X)$, $\Phi_{10}(X)$, and $\Phi_{12}(X)$.

(d) Use part (b) to find Φ_{p^n}, where p is a prime integer.

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Question 5. Let F be a field of characteristic $p > 0$ and E/F an extension with $[E : F]$ relatively prime to p. Show that E/F is a separable extension.

Question 6. Prove that $d|n$ if and only if $X^d - 1$ divides $X^n - 1$.

Question 7. Prove that the property of normality is not transitive by finding a tower of fields $F \subset E \subset K$ with E/F normal, K/E normal, but K/F not normal.

Question 8. Let K be a field of char $p > 0$.

(a) Show that $f : K \to K$ defined by $x \mapsto x^p$ is a field homomorphism. It is called the Frobenius homomorphism.

(b) Show that K is a perfect field if and only if f is an automorphism.