Math 210B Homework 6

Question 1. Let K be a finite extension of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[X]$ with a root in K splits completely in $K[X]$.

Question 2. Let K_1 and K_2 be finite extensions of F contained in a larger field L, and assume both are splitting field over F.

(a) Prove that their composite K_1K_2 is a splitting field over F.

(b) Prove that $K_1 \cap K_2$ is a splitting field over F. (Hint: Use problem 1).

Question 3. Determine the splitting field and its degree over $\mathbb{Q}$ for $X^4 + X^2 + 1$.

Question 4. Let K/F be a field extension, $\alpha \in K$. Show that if for some non-constant polynomial $f \in F[X]$ the element $f(\alpha)$ is algebraic over F, then α is also algebraic over F.

Question 5. Prove that it is impossible to construct the regular 9-gon using ruler and compass. (Hint: This is equivalent to constructing $\cos(\frac{2\pi}{9})$).

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Question 6. Let K be the splitting field for $X^3 - 2$. Find $[K : \mathbb{Q}]$.

Question 7. Let $K = \mathbb{Q}(\sqrt{2i})$. Find $[K : \mathbb{Q}]$.

Question 8. Suppose K/F is a field extension of degree p, for p a prime. Show that if E is any field such that $F \subset E \subset K$, then either $E = F$ or $E = K$.

Question 9. Use the fact that $\alpha = 2\cos(\frac{2\pi}{5})$ satisfies the equation $X^2 + X - 1 = 0$ to conclude the regular 5-gon is constructible by ruler and compass.

Question 10. Use the fact that $\beta = 2\cos(\frac{2\pi}{7})$ satisfies the equation $X^3 + X^2 - 2X - 1 = 0$ to show that the regular 7-gon is not constructible by ruler and compass.