Math 210B Homework 5

Question 1. Let R be noetherian and $I \subset R$ an ideal. An I*-filtration* of an R-module M

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$$

is a sequence of submodules such that $I \cdot M_n \subset M_{n+1}$.

(a) Show that if R is noetherian then so is the (graded) ring $S := R \oplus I \oplus I^2 \oplus \ldots$ (make the ring structure of S explicit).

(b) Let $\{M_n\}_{n \geq 0}$ be an I-filtration of M. Show that $M_S := M \oplus M_1 \oplus M_2 \oplus ...$ has a natural structure of (graded) S-module.

An I-filtration as above is *(I-)stable* if there exists n_0 such that $M_{n+1} = I \cdot M_n$ for all $n \geq n_0$.

(c) Show that the above M_S is finitely generated over S if and only if M is finitely generated over R and the I-filtration $\{M_n\}_{n \geq 0}$ is stable.

Question 2 (**Artin-Rees Lemma**)**.** Let R be a noetherian ring and $I \subset R$ be an ideal. Let M be a finitely generated R-module and $N \subset M$ be a submodule.

(a) Let $\{M_n\}_{n \geq 0}$ be a stable I-filtration of M. Show that $N_n := N \cap M_n$ defines a stable I-filtration of N.

(b) Show that there exists an integer k such that $I^n M \cap N = I^{n-k}(I^k M \cap N)$ for all $n \geq k$.

Question 3.

(a) Prove that the $\mathbb{Z}$-module $\mathbb{Z}$ is noetherian but not artinian.

(b) Prove that the $\mathbb{Z}$-module $M = \cup_{n=1}^{\infty}(p^{-n}\mathbb{Z}/\mathbb{Z}) \subset \mathbb{Q}/\mathbb{Z}$ is artinian but not noetherian.

Question 4.

(a) Find an example of a nontrivial field extension K/F with isomorphic fields K and F.

(b) Find an example of an infinite algebraic extension.

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Question 5.

(a) Let K/F be a field extension, $\alpha, \beta \in K$. Show that if $[F(\alpha) : F]$ and $[F(\beta) : F]$ are relatively prime, then $[F(\alpha, \beta) : F] = [F(\alpha) : F] \cdot [F(\beta) : F]$.

(b) Let K/F be a field extension, $\alpha, \beta \in K$. Prove that the extension $F(\alpha, \beta)/F(\alpha+\beta, \alpha\beta)$ is algebraic.

Question 6. Prove that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.

Question 7. Let X be a variable over a field F, and $Y = \frac{X^2}{X-1}$. Find $[F(X) : F(Y)]$.

Question 8. Let K/F be an algebraic extension and let R be a *ring* contained in K and containing F. Show that R is a subfield of K containing F.