Math 210B Homework 4

Question 1. Let F be a field and let $A \in \mathrm{M}_n(F)$. Consider the ring $R = F[X]$ and view A as an element of $\mathrm{M}_n(R)$. Consider the matrix $X \cdot I_n - A$ in $\mathrm{M}_n(R)$ as a linear endomorphism of R^n. Find π such that the following sequence of R-modules is exact

$$0 \longrightarrow R^n \xrightarrow{(X \cdot I_n - A)} R^n \xrightarrow{\pi} V \longrightarrow 0$$

where $V = F^n$ is the R-module characterized by $X \cdot v = A \cdot v$ for any $v \in F^n$.

Question 2. Let $R = \mathbb{Q}[X]$ and let M be the submodule of R^2 generated by the elements $(1 - 2x, -x^2)$ and $(1 - x, x - x^2)$. Express R^2/M as a direct sum of cyclic modules (i.e., modules generated by one element).

Question 3. Let R be a commutative ring with unit and let M be a finitely generated R-module. Suppose $f : M \longrightarrow M$ is an R-linear endomorphism which is surjective. Show that f is an isomorphism. (Hint: View M as an $R[X]$-module and use Nakayama).

Question 4. Prove there are no 3×3 matrices A over $\mathbb{Q}$ with $A^8 = I$ but $A^4 \neq I$.

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Question 5. Let F be a field. Prove that every $A \in \mathrm{M}_n(F)$ is similar to its transpose.

Question 6. Find the Jordan canonical form of the linear operator in $\mathbb{C}^3$ given by

$$\begin{pmatrix} 5 & 4 & 1 \\ -1 & 0 & 0 \\ -3 & -4 & 1 \end{pmatrix}. \tag{1}$$

Question 7. Let $A \in \mathrm{M}_n(F)$. Prove that the minimal polynomial of A has the same irreducible divisors as the characteristic polynomial of A.

Question 8. Suppose M is a finitely generated, artinian R-module, where R is a commutative ring with 1. Suppose $f : M \longrightarrow M$ is an R-module homomorphism which is injective. Show that f is an isomorphism. (Hint: Consider a descending chain of cokernels.)