Math 210B Homework 2

Question 1.

(a) Let $R \to S$ be a homomorphism of commutative rings. Let $I \subset R[X_1, ..., X_n]$ be an ideal. Show that there is an isomorphism of S-modules (actually of rings, see Q. 9) :

$$S \otimes_R (R[X_1, ..., X_n]/I) \cong S[X_1, ..., X_n]/I \cdot S[X_1, ..., X_n]\,. \tag{1}$$

(b) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-vector spaces (actually, as rings). Deduce that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are two non-isomorphic $\mathbb{R}$-vector spaces.

Question 2. Let $R = \mathbb{Z}[X]$. Show that there is a short exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\cdot X} R \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0\,.$$

What is the structure of R-module on $\mathbb{Z}$ and what is π? Does the above sequence split? Does it split as a sequence of abelian groups?

Question 3. Let $\{M_i\}_{i \in I}$ be a collection of R-modules.

(a) Show that the coproduct $\coprod_{i \in I} M_i$ is projective if and only if each M_i is.

(b) Show that the product $\prod_{i \in I} M_i$ is injective if and only if each M_i is.

(c) Prove that $M \oplus N$ is a projective (resp. injective) if and only if both M and N are.

Question 4. Show that projective modules are flat. [Hint: Show that free modules are flat.]

Question 5. Let R be a commutative ring and M an R-module. Let $M^* = \mathrm{Hom}_R(M, R)$ be the *dual* R-module. Recall the R-module structure on M^*.

(a) Show that there is a natural R-homomorphism $\iota_M : M \to M^{**}$ as in linear algebra.

(b) Show that ι_M is an isomorphism when M is finitely generated projective.

(c) Let $\mathrm{Bilin}_R(M)$ be the R-module of R-bilinear maps $\beta : M \times M \longrightarrow R$. Show that there is an isomorphism between $\mathrm{Bilin}_R(M)$ and $\mathrm{Hom}_R(M, M^*)$.

(d) Show that under the isomorphism of (c), the endomorphism $(-)^t$ of $\mathrm{Bilin}_R(M)$ defined by $\beta^t(x, y) = \beta(y, x)$ corresponds to the endomorphism $(-)^t$ of $\mathrm{Hom}_R(M, M^*)$ defined by $\varphi^t := \varphi^* \circ \iota_M \,: M \xrightarrow{\iota_M} M^{**} \xrightarrow{\varphi^*} M^*$. Both generalize *transposition* of linear algebra.

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Question 6. Prove that in the category of vector spaces over a field k, any object is both projective and injective.

Question 7. Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

Question 8. Prove that every $\mathbb{Z}/6\mathbb{Z}$-module is projective and injective. Find a $\mathbb{Z}/4\mathbb{Z}$-module that is neither projective nor injective.

Question 9. Let R be a commutative ring and let A and B be two R-algebras (a ring C is an R-*algebra* if there exists a ring homomorphism $R \to C$ whose image is in the center of C; one can then view C as an R-module as usual.) Show that $A \otimes_R B$ becomes a ring such that $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$.

Prove for instance that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ as rings, where $d = \gcd(m, n)$. Show also that the isomorphism (1) in Question 1 is an isomorphism of rings.

Question 10. Find a flat module which is not projective.