Math 210B Homework 10

Question 1. Let K/F be a finite field extension. The trace $\mathrm{tr}(x)$ of an element $x \in K$ is the trace of the linear operator ℓ_x on the F-vector space K defined by $\ell_x(y) = xy$.

(a) Let $x \in K$ be an element such that $x^2 \in F$ but $x \notin F$. Prove that $\mathrm{tr}(x) = 0$.

(b) Let $n_1, n_2, ..., n_k$ be positive rational numbers such that

$$\sqrt{n_1} + \sqrt{n_2} + \cdots + \sqrt{n_k} \in \mathbb{Q}. \tag{1}$$

Prove that $\sqrt{n_i} \in \mathbb{Q}$ for all $i = 1, ..., k$.

Question 2. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt[7]{5})$.

(a) Prove that K has only one subfield $F \subset K$ such that $[F : \mathbb{Q}] = 2$.

(b) Find all subfields of K.

(c) Find an element $u \in K$ such that $K = \mathbb{Q}(u)$.

(d) Describe all elements $u \in K$ such that $K = \mathbb{Q}(u)$.

Question 3. Prove that the cyclotomic polynomial $\Phi_n(X)$ is irreducible for every n. [One possibility: Consider the prime factorization of $X^n - 1$ in $\mathbb{Z}[T]$, hence in $\mathbb{Q}[T]$. Let $\Psi \in \mathbb{Z}[X]$ be the minimal polynomial of $\zeta = e^{\frac{2\pi i}{n}}$ and let $\chi(X) \in \mathbb{Z}[X]$ such that $X^n - 1 = \Psi(X) \cdot \chi(X)$. Show that for any prime p not dividing n, we have $\Psi(\zeta^p) = 0$ by showing that $\chi(\zeta^p) = 0$ would imply that Ψ divides $\chi(X^p)$ and yield a contradiction in $\mathbb{Z}/p[X]$. Deduce that Ψ is the minimal polynomial of every primitive nth root of unity. Using Question 4 of HW 7, deduce that $\Psi = \Phi_n$.]

Question 4. Recall that the group S_n acts on a polynomial $f(T_1, \ldots, T_n) \in A[T_1, \ldots, T_n]$ by $\sigma f(T_1, \ldots, T_n) = f(T_{\sigma(1)}, \ldots, T_{\sigma(n)})$ for all $\sigma \in S_n$. We say f is *symmetric* if $\sigma f = f$ for all $\sigma \in S_n$. We define the *weight* of a monomial $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ to be $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ and we define the weight of a polynomial $g(X_1, \ldots, X_n)$ to be the maximum of the weights of the monomials in g.

Let $f(T_1, \ldots, T_n) \in A[T_1, \ldots, T_n]$ be symmetric of degree d. Prove that there exists a polynomial $g(X_1, \ldots, X_n)$ of weight $\leq d$ such that

$$f(T_1, \ldots, T_n) = g(s_1, \ldots, s_n), \tag{2}$$

where $s_1, \ldots, s_n$ are the symmetric polynomials. (Solution: See §IV.6 in Lang.)

Question 5. (a) Let $P(T) \in \mathbb{Q}[T]$ be irreducible of degree p, a prime. Suppose that P has precisely two non-real roots. Show that the Galois group of P is S_p.

(b) What is the Galois group of $P(T) = T^5 - 4T + 2$?

(Solution: See §VI.2 in Lang.)