Math 210B Homework 1

Question 1. Let R be a ring.

(a) Let $I \subset R$ be a two-sided ideal. Show that there is an isomorphism between the category of left R/I-modules and that of left R-modules such that $IM = 0$.

(b) Let $S \subset R$ be a central multiplicative subset. As above, determine an isomorphism between the category of left $S^{-1}R$-modules and some subcategory of left R-modules.

Question 2.

(a) Show that if the free R-modules R^n and R^m over a non-zero commutative ring R are isomorphic, then $n = m$. [Hint: Free modules remain free under extension of scalars.]

(b) Show that (a) can fail for non-commutative rings.

Question 3. Prove that a (left) R-module generated by one element is isomorphic to R/I, where I is some (left) ideal of R.

Question 4. Determine all $\mathbb{Z}[X]$-modules structures on the group $\mathbb{Z}/5\mathbb{Z}$.

Question 5. Determine all integers $n > 0$ such that $\mathbb{Z}/n\mathbb{Z}$ has a $\mathbb{Z}[i]$-module structure.

Question 6. Show that a submodule of a free module needs not be free.

Question 7. Show that the forgetful functor $U : R\text{–}\mathsf{mod} \longrightarrow \mathsf{Sets}$ has a left adjoint.

Question 8. Show that a commutative ring for which all modules are free is a field. Is this true in the non-commutative case?

Question 9. How is $M = \mathrm{M}_{m\times n}(R)$ an A-B-bimodule for $A = \mathrm{M}_m(R)$ and $B = \mathrm{M}_n(R)$?

Question 10. *Adjunction between tensor product and* Hom*-functor* : Let A, B, C, D be rings, let L be an A-B-bimodule, let M be an B-C-bimodule and let N be a D-C-module. Then there is an isomorphism of D-A-bimodules

$$\mathrm{Hom}_C(L \otimes_B M, N) \cong \mathrm{Hom}_B(L, \mathrm{Hom}_C(M, N)). \tag{1}$$

Make all module structures explicit. State the special case $A = D = \mathbb{Z}$. With the same left-hand side as in (1), rephrase the above "with L moving to the right instead of M".