MATH 210A HOMEWORK 9

Problem 1.

(a) Let D be a domain and $c \in D$. Show that a polynomial $f(X) = \sum_{i=0}^{n} a_i X^i \in D[X]$ is irreducible in $D[X]$ if and only if $f(X-c) = \sum_{i=0}^{n} a_i (X-c)^i \in D[X]$ is irreducible.

(b) For each prime integer p, prove that the cyclotomic polynomial $X^{p-1} + X^{p-2} + \cdots + X + 1$ is irreducible in $\mathbb{Z}[X]$. (Hint: Use Eisenstein and part (a).)

Problem 2.

(a) Prove that $X^2 + Y^2 - 1$ is irreducible in $\mathbb{Q}[X, Y]$.

(b) Prove that $X^5 + Y^5 + Z^5$ is irreducible in $\mathbb{C}[X, Y, Z]$.

(c) Determine whether or not the polynomial $2X^5 - 6X + 6$ is irreducible in the following rings: $\mathbb{Z}[X]$, $(S^{-1}\mathbb{Z})[X]$ where $S = \{2^n \mid n \geq 0\}$, $\mathbb{Q}[X]$, $\mathbb{R}[X]$, and $\mathbb{C}[X]$.

(d) Determine all $n \in \mathbb{Z}$ such that the polynomial $X^3 + nZ + 2$ is irreducible in $\mathbb{Q}[X]$.

Problem 3.

(a) Given a polynomial $f \in R[X]$ and $c \in R$, show that $f \equiv f(c) \bmod (X - c)$.

(b) Show that f has a root at c if and only if $f(c) = 0$.

(c) When R is a domain, show that $f \in R[X]$ of degree d has at most d roots.

(d) When R is a domain, show that f has a multiple root at c iff $f(c) = 0$ and $f'(c) = 0$ (where f' is the usual derivative).

Problem 4. Show that the ring $R = \mathbb{Z}[x_1, x_2, \ldots] = \mathbb{Z}[x_i]_{i \in \mathbb{N}}$ is a UFD. Is it noetherian?

Problem 5.

(a) Suppose R is noetherian and $I \subset R$ is an ideal of R. Prove that R/I is noetherian.

(b) Suppose R is artinian and $I \subset R$ is an ideal of R. Prove that R/I is artinian.

(c) If $R[x]$ is noetherian, does R necessarily have to be noetherian as well?

Problem 6.

(a) Let R be a noetherian ring and S a central multiplicative subset. Prove that $S^{-1}R$ is noetherian.

(b) Let R be an artinian ring and S a central multiplicative subset. Prove that $S^{-1}R$ is artinian.

Problem 7. Prove that any artinian integral domain is a field.

Problem 8. Give two examples of noetherian rings that are not artinian.

Problem 9. Let R be the ring of 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$. Prove that R is right noetherian but not left noetherian.

Problem 10. Let R be the ring of 2×2 matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ such that $a \in \mathbb{Q}$ and $b, c \in \mathbb{R}$. Prove that R is right artinian but not left artinian.