MATH 210A HOMEWORK 8

Problem 1. Let R be a domain and $S \subset R$ be a multiplicative subset with $0 \notin S$. Consider the localization $S^{-1}R$.

(a) Prove that if R is a PID then $S^{-1}R$ is also a PID.
(b) Is the same true for UFD instead of PID?

Problem 2. Let R be the ring $\{a+b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$, with norm $\mathcal{N}(a+b\sqrt{10}) = a^2-10b^2$.

(a) Show that $\mathcal{N}(u)\mathcal{N}(v) = \mathcal{N}(uv)$ for all $u, v \in R$ and that $\mathcal{N}(u) = 0$ if and only if $u = 0$.
(b) Prove that $u \in R$ is a unit if and only if $\mathcal{N}(u) = \pm 1$.
(c) Show that $2, 3, 4+\sqrt{10}$ and $4-\sqrt{10}$ are all irreducible elements of R.
(d) Use these elements to deduce that R is not a UFD.

Problem 3. Show that a prime p in $\mathbb{Z}$ remains prime in $\mathbb{Z}[i]$ if and only if p is congruent to 3 mod 4.

Problem 4. Let R be a commutative ring and let $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$.

(a) Prove that the polynomial f is nilpotent if and only if all a_i are nilpotent in R.
(b) Show that f is a zero divisor in $R[X]$ if and only if there exists a non-zero $b \in R$ such that $ba_n = \cdots = ba_1 = ba_0 = 0$.
(c) Show that f is invertible if and only if a_0 is invertible and all a_i are nilpotent for $i \geq 1$.
(d) If R is a domain, what is $(R[X])^\times$?

Problem 5. Let R and S be two commutative rings.

(a) Let $\phi : R \to S$ be a ring homomorphism. Fix elements $s_1, \ldots, s_n \in S$. Show that the map $R[X_1, \ldots, X_n] \to S$ given by $f(X_1, \ldots, X_n) \mapsto (\phi(f))(s_1, \ldots, s_n)$ is a ring homomorphism (sometimes called the *evaluation homomorphism*).
(b) Describe all ring homomorphisms from $R[X_1, \ldots, X_n] \to S$.
(c) Prove that part (a) is false if we remove the commutative assumption.
(d) Is the ring $\mathbb{Z}[X_1, \ldots, X_n]$ special in anyway?

Problem 6. Let R be a ring.

(a) Let a be a nilpotent element of R. Prove that $1 + a$ is invertible in R.
(b) Suppose that u is invertible and a is nilpotent. What do you need to conclude that $u + a$ is invertible?