MATH 210A HOMEWORK 7

Problem 1. Let R be a commutative ring equipped with a filtration of ideals satisfying $\bigcap_{n\geq 1} I^{(n)} = (0)$.

(a) Let $R^{\mathbb{N}}$ denote the set of sequences in R with the pointwise ring structure. Show that the set $\mathfrak{C}(R)$ of Cauchy sequences forms a subring of $R^{\mathbb{N}}$ and that the set $\mathfrak{C}_0(R)$ of sequences converging to 0 forms an ideal of $\mathfrak{C}(R)$.

(b) Establish an isomorphism of rings $\mathfrak{C}(R)/\mathfrak{C}_0(R) \simeq \lim_{n\geq 1} R/I^{(n)}$.

(c) Show that the $\hat{I}^{(\cdot)}$-adic topology on $\lim_{n\geq 1} R/I^{(n)}$ is the same as the topology one gets by giving each $R/I^{(n)}$ the discrete topology and taking the limit $\lim_{n\geq 1} R/I^{(n)}$ in **TopRings**.

(d) Recall that the metric space completion of a metric space X is the universal uniformly continuous map from X to a complete metric space. Establish that the $I^{(\cdot)}$-adic completion of R satisfies this universal property. [Note: A continuous ring homomorphism between metric rings is automatically uniformly continuous.]

Problem 2. Equip a commutative ring R with the I-adic topology, for some ideal $I \subset R$.

(a) Show that a series $\sum_{i=0}^{\infty} x_i$ is Cauchy if and only if $\lim_{i\to\infty} x_i = 0$.

(b) Is the above valid for any topological ring?

(c) Show that any element of the completion $R_I^{\wedge}$ is equal to a sum $\sum_{i=0}^{\infty} x_i$ with $x_i \in I^i$.

(d) Show that every p-adic integer $a \in \mathbb{Z}_p^{\wedge}$ can be written as the sum of an infinite series $a = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots$ with $0 \leq a_i \leq p-1$ for all $i \geq 0$.

(e) Given $a \in \mathbb{Z}_p^{\wedge}$ as in (d) above, are the coefficients a_i unique?

(f) Compute $a \in \mathbb{Z}_p^{\wedge}$ as in (d) above when $a_i = p-1$ for all $i \geq 0$.

(g) Show that for any $b \in \hat{I}^{(1)}$, the element $1+b$ is invertible in $\hat{R}$. Describe $(1+b)^{-1}$.

(h) Show that x is invertible in $\hat{R}$ if and only if $\pi_1(x) \in R/I$ is invertible.

(i) Determine when an element of $k[\![X]\!]$ is invertible (for k a commutative ring).

(j) Are there continuous ring homomorphisms $\mathbb{Z}[\![X]\!] \to \mathbb{R}$?

(k) Are there continuous ring homomorphisms $\mathbb{Z}_p^{\wedge} \to \mathbb{R}$?

Problem 3. Find a solution of $x^2 = 10$ in the ring of 3-adic integers $\mathbb{Z}_3^{\wedge}$.

Problem 4. Let k be a field. Show that $k[\![X]\!]$ is principal. [Hint: Show more.]

Problem 5. Let R be a principal domain and let $a, b \in \mathbb{R}$ be non-zero. Show that $Ra + Rb = R \cdot \gcd(a,b)$ and $Ra \cap Rb = R \cdot \mathrm{lcm}(a,b)$. Is this true in any UFD?

Problem 6. Let R be a Euclidean domain with "Euclidean function" $\phi : R \setminus \{0\} \to \mathbb{N}$.

(a) Show that one can replace ϕ by another Euclidean function $\nu : R \setminus \{0\} \to \mathbb{N}$ such that:

$$\nu(a) \leq \nu(ab) \qquad \text{for all } b \in R \setminus \{0\}.$$

This (unnecessary) condition is often included in the definition of Euclidean.

(b) Assume that the Euclidean function satisfies the above condition. Then show that $u \in R$ is invertible if and only if $\phi(u) = \phi(1)$.

Problem 7. Show that the ring of Gaussian integers $\mathbb{Z}[i]$ is Euclidean. What are the units in this ring?