MATH 210A HOMEWORK 6

Problem 1. Describe all ring homomorphisms from $\mathbb{R}$ to $\mathbb{R}$.

Problem 2.

(a) Let $R = \mathbb{Z}$, $I_1 = 6\mathbb{Z}$ and $I_2 = 15\mathbb{Z}$. Show that the canonical map $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ is not surjective.

(b) Let n be an integer with prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Prove that

$$\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\alpha_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{\alpha_k}\mathbb{Z}.$$

as rings. Use this to establish the following isomorphism on the group of units

$$(\mathbb{Z}/n\mathbb{Z})^\times \simeq (\mathbb{Z}/p_1^{\alpha_1}\mathbb{Z})^\times \times (\mathbb{Z}/p_2^{\alpha_2}\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_k^{\alpha_k}\mathbb{Z})^\times.$$

Problem 3. Let R be a commutative ring, and I an ideal of R. Define the *radical* of I, denoted $\sqrt{I}$, to be

$$\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n > 0\}.$$

(a) Prove that $\sqrt{I}$ is an ideal of R containing I.

(b) Let I and J be ideals of R. Prove that $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(c) Prove that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

(d) Prove that $\sqrt{I} = R$ if and only if $I = R$.

Problem 4. Let R be a commutative ring. We call $\mathcal{N} := \sqrt{(0)}$ the *nilradical* of R.

(a) We say a ring is *reduced* if the only nilpotent element is 0. Prove that $R/\mathcal{N}$ is reduced.

(b) Prove that if J is an ideal of R with R/J reduced, then $J \supset \mathcal{N}$.

Problem 5.

(a) Formalize the meaning of the following statement: "Inverting multiplicative subsets commutes with taking radicals." Prove it.

(b) Formalize the meaning of the following statement: "Inverting multiplicative subsets commutes with quotienting by ideals." Prove it.

Problem 6. Let R be an integral domain with field of fractions F. Show that if R' is an integral domain with $R \subset R' \subset F$, then the field of fractions of R' is isomorphic to F. Give an example with $R \subsetneq R'$.

Problem 7. Let R be a ring and $S \subset R$ be a central multiplicative subset.

(a) Prove that S consists of invertible elements if and only if $R \simeq S^{-1}R$.

(b) Give necessary and sufficient conditions for $S^{-1}R$ to be non-zero.

(c) Give necessary and sufficient conditions for $R \to S^{-1}R$ to be injective.

(d) Let $R = \mathbb{Z}/6\mathbb{Z}$ and $S = \{1, 2, 4\}$. Prove that S is multiplicative, and determine $S^{-1}R$.

Problem 8. Let R be a ring and $S \subset R$ a central multiplicative subset.

(a) For every ideal $I \subset R$ show that $S^{-1}I := \{\frac{a}{s} \mid a \in I, s \in S\}$ is an ideal of $S^{-1}R$.

(b) Show that every ideal of $S^{-1}R$ is of the form $S^{-1}I$ as above. Is the I unique?

(c) Show that the operation $I \mapsto S^{-1}I$ commutes with sum of ideals, intersection and product.
(d) Show that the localization of a principal domain is principal.
(e) Describe the ideals of $\mathbb{Z}[1/a] = S^{-1}\mathbb{Z}$ where $S = \{1, a, a^2, a^3, \ldots\}$.
(f) Describe the ideals of $\mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$ where $S = \{a \in \mathbb{Z} \mid p \nmid a\}$ for p a prime.

Problem 9. Let R be a ring and let $S \subset \mathbb{Z}$ be a multiplicative subset of the integers. Show that there is at most one ring homomorphism $S^{-1}\mathbb{Z} \to R$.

Problem 10. Let R be a ring and let $n \in \mathbb{Z}$ be an integer. Show that there is at most one ring homomorphism $\mathbb{Z}/n\mathbb{Z} \to R$.

Problem 11. Let R be a ring and $S \subset R$ a central multiplicative subset. Prove that the canonical map $R \to S^{-1}R$ is an epimorphism in the category of rings.

Problem 12. Let R be ring. An element $e \in R$ is said to be *idempotent* if $e^2 = e$. A *central idempotent* is an idempotent that lies in the center of the ring.

(a) What are the "trivial" idempotents?
(b) Show that if R has a central idempotent e, then $R \simeq Re \times R(1-e)$ as rings. Conversely, if $R \simeq R_1 \times R_2$, show that R has a central idempotent yielding this decomposition.
(c) How many idempotents are there in an integral domain?
(d) How many idempotents are there in $\mathbb{Z}/p^m\mathbb{Z}$, for p a prime and $m \geq 1$?
(e) How many idempotents are there in $\mathbb{Z}/n\mathbb{Z}$, for n an integer?

Problem 13. (Lifting of idempotents modulo a nilideal). Let R be a ring and I an ideal such that all the elements of I are nilpotent. Show that if $\bar{e}$ is an idempotent in R/I then there is an idempotent preimage for $\bar{e}$ in R.