MATH 210A HOMEWORK 5

Problem 1.

(a) Determine all subgroups of A_4. Show that A_4 has no subgroup of order 6.
(b) Determine all normal subgroups of A_4.
(c) Does there exist a nontrivial group action of A_4 on a set of two elements?

Problem 2.

(a) Prove that every group of order p^2 (for p a prime) is abelian.
(b) Let G be a group with $|G| = p \cdot q$ for p and q two prime numbers, $q > p$ and $q \not\equiv 1 \pmod{p}$. Prove that $G \simeq \mathbb{Z}/pq\mathbb{Z}$.
(c) Is there a group G such that $G/Z(G)$ has order 143?

Problem 3. Prove that any group of order $2k$, where k is odd, has a normal subgroup of index 2. (Hint: Let G act on itself by left translation, and G has an element of order 2.)

Problem 4. Classify all groups of order 154. (Hint: First use Problem 3.)

Problem 5.

(a) Exhibit all composition series for the quaternion group Q_8.
(b) Exhibit all composition series for the dihedral group D_8.

Problem 6.

(a) Prove that there are no simple groups of order 132.
(b) Prove that there are no simple groups of order 6545.
(c) How many elements of order 7 must there be in a simple group of order 168?

Problem 7.

(a) Find all finite groups that have exactly two conjugacy classes.
(b) Find all finite groups that have exactly three conjugacy classes.

Problem 8. Consider $G := \mathrm{Aut}_{\mathbf{Sets}}(\mathbb{N})$. For $\sigma \in G$, define as usual its *fixed set* by $\mathbb{N}^\sigma := \{a \in \mathbb{N} \mid \sigma(a) = a\}$ and let $M(\sigma) := \mathbb{N} - \mathbb{N}^\sigma$ be the set moved by σ.

(a) Show that $S_\infty := \{\sigma \in G \mid M(\sigma) \text{ is finite}\}$ is a normal subgroup of G.
(b) Prove that $S_\infty \simeq \mathrm{colim}_{n\in\mathbb{N}} S_n$ under explicit inclusions $S_n \hookrightarrow S_{n+1}$.
(c) Let A be the subgroup of S_∞ consisting of those σ that act as an even permutation on $M(\sigma)$. Prove that $A = \mathrm{colim}_{n\in\mathbb{N}} A_n$.
(d) Prove that A is an infinite simple group.
(e) Find an example of another infinite simple group. (Hint: Think of groups of matrices.)

Problem 9. Let $H = \mathbb{Z}$ and let $K = \mathbb{Z}/2\mathbb{Z}$. Discuss the structure of the group $H \rtimes_\phi K$ when ϕ is the nontrivial homomorphisms $\phi : K \to \mathrm{Aut}(H)$. This group is known as D_∞, the *infinite dihedral group*. Prove that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \simeq D_\infty$.

Problem 10.

(a) A group is said to be *nilpotent* if it admits a normal tower
$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G$$
with the property that $H_{i+1}/H_i \subset Z(G/H_i)$ for each i. The minimum possible length of such a "central tower" is called the *nilpotency class* of G.

(b) The *upper central series* of a group G is a sequence of subgroups defined by setting $Z_0(G) = 1$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G)$ to be the subgroup of G containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Prove that G is nilpotent if and only if $Z_c(G) = G$ for some $c \in \mathbb{N}$.

(c) The *lower central series* of a group G is a sequence of subgroups defined by setting $G_0 = G$, $G_1 = [G, G]$, and $G_{i+1} = [G, G_i]$. Prove that G is nilpotent if and only if $G_c = 1$ for some $c \in \mathbb{N}$.

(d) Observe that the nilpotency class of a nilpotent group is equal to the length of its upper central series and is also equal to the length of its lower central series.

(e) Observe that the trivial group is the unique group of nilpotency class 0 and that the non-trivial abelian groups are exactly the groups of nilpotency class 1.

(f) Let $1 \to N \to G \to H \to 1$ be a *central* extension (i.e. $N \subset Z(G)$). Show that G is nilpotent if and only if N and H are, if and only if H is.

(g) Can one remove "central" in the previous statement?

(h) Show that any p-group is nilpotent.

(i) Show that the cartesian product of a finite number of nilpotent groups is nilpotent.

(j) Show that nilpotent implies solvable, but that the converse is false.

Problem 11. Let G be a finite group. Prove that the following are equivalent:

(a) G is nilpotent.

(b) Every Sylow subgroup of G is normal.

(c) For every prime p dividing $|G|$, there exists a unique p-Sylow subgroup of G.

(d) G is the direct product of its Sylow subgroups.

(e) G is a direct product of p-groups.

Hint: (1) If P is a Sylow subgroup of a finite group G then $N_G(N_G(P)) = N_G(P)$; (2) If G is a nilpotent group and H is a proper subgroup of G then H is properly contained in $N_G(H)$.

Problem 12. Let $H_3(\mathbb{Z})$ be the group of 3×3 integer matrices of the form $\left(\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Prove that $H_3(\mathbb{Z})$ is a nilpotent group. This is an example of an infinite nilpotent group.