MATH 210A HOMEWORK 4

Problem 1. (The Alternating Group). We define the group A_n to be the kernel of the signature homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\}$ and we call it the *alternating group on n letters*.

(a) Show that A_n is a normal subgroup in S_n and compute its order.
(b) Show that $S_n = A_n \rtimes \mathbb{Z}/2$.
(c) Show that, for $n \geq 3$, A_n is generated by the 3-cycles $(i\ j\ k)$ in S_n. (Hint: A_n is generated by products of two transpositions.)
(d) Show that A_n, for $n \geq 4$, and S_n, for $n \geq 3$, have trivial centers.

Problem 2. Let G be a group. The *commutator* of two elements $g, h \in G$ is $[g, h] := g \cdot h \cdot g^{-1} \cdot h^{-1}$. The *commutator subgroup* $[G, G]$ is the subgroup generated by all commutators.

(a) Prove that $[S_n, S_n] = A_n$ for $n \geq 3$. (Hint: Recall that A_n is generated by the 3-cycles.)
(b) Prove that $[GL_n(\mathbb{C}), GL_n(\mathbb{C})] = SL_n(\mathbb{C})$ for $n \geq 3$. (Hint: Recall that $SL_n(\mathbb{C})$ is generated by the elementary matrices $E_{ij}(\lambda)$ which have 1's on the diagonal, λ in the i, j spot (where $i \neq j$), and 0 everywhere else.)
(c) Show that the commutator subgroup gives rise to a functor **Grps** $\to$ **Grps**, and that $[G, G]$ is a characteristic subgroup of G (invariant under any automorphism of G).

Problem 3. Find a left adjoint functor to the inclusion functor from **Ab** to **Grps**.

Problem 4. Let $n \geq 2$ be an integer. Show that the only subgroup of index 2 in S_n is A_n.

Problem 5. Let H and K be subgroups of finite index of a group G. Then $[G : H \cap K]$ is finite and $[G : H \cap K] \leq [G : H][G : K]$. Furthermore, $[G : H \cap K] = [G : H][G : K]$ if and only if $G = HK$.

Problem 6.

(a) Show that Q_8 is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.
(b) Recall that D_8, the dihedral group of order 8, is a split extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.
(c) Show that a non-abelian group G of order 8 is necessasrily an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/4\mathbb{Z}$.
(d) Show that a non-abelian group of order 8 is isomorphic to either D_8 or Q_8.
(e) Describe the isomorphism clases of groups of order 8.

Problem 7. (The 5-Lemma). Consider the following diagram of groups:

$$\begin{tikzcd}
G_1 \arrow[r] \arrow[d, "\simeq"] & G_2 \arrow[r] \arrow[d, "\simeq"] & G_3 \arrow[r] \arrow[d, "\alpha"] & G_4 \arrow[r] \arrow[d, "\simeq"] & G_5 \arrow[d, "\simeq"] \\
H_1 \arrow[r] & H_2 \arrow[r] & H_3 \arrow[r] & H_4 \arrow[r] & H_5
\end{tikzcd}$$

where each square commutes and each row is an exact sequence (ker = im at each group).

(a) Prove that the morphism $\alpha : G_3 \to H_3$ is an isomorphism as well.
(b) Can we weaken the hypotheses on the (external) vertical homomorphisms?