MATH 210A HOMEWORK 3

Problem 1. Let A be a semigroup (that is, a set with an associative law $a \cdot b$).

(a) Suppose A has a left identity element $e_L \in A$ (that is, $e_L \cdot a = a$ for each $a \in A$). Suppose further that each element $a \in A$ has a left inverse. Prove that A is a group.

(b) Suppose now that A has a left identity and every element has a right inverse. Is this enough to conclude that A is a group?

Problem 2. Let G be a cyclic group.

(a) Describe all subgroups of G.

(b) Find all automorphisms of G.

Problem 3. Provide a group G, a subgroup $H \leq G$ and an element $g \in G$ such that $gHg^{-1} \subset H$ but without equality. Does g belong to the normalizer of H in G?

Problem 4.

(a) Show that if $g^2 = e$ for every g in a group G, then G is abelian.

(b) Show that every subgroup of index $p = 2$ is normal.

(c) Let p be an odd prime. Find a group with a non-normal subgroup of index p.

(d) Prove that if G is a finite group of even order, then G contains an element a such that $a \neq e$ but $a^2 = e$.

Problem 5.

(a) Determine the order of the symmetric group S_n.

(b) Prove that S_n is generated by all the transpositions.

(c) Prove that S_n is, in fact, generated by the transpositions $(1\ 2), (1\ 3), \ldots, (1\ n)$.

(d) Prove that S_n can be generated by the transposition $(1\ 2)$ and the n-cycle $(1\ 2\ \cdots\ n)$.

Problem 6. Show that in any expression of a given permutation as a product of transpositions, the number of transpositions is always odd or always even. Use this to define the "signature" homomorphism $\operatorname{sgn} : S_n \to \{\pm 1\}$. The kernel of this group homomorphism is the alternating group A_n.

Problem 7. Let D_8 be the group of isometries of a square (distance-preserving bijections).

(a) Show that it is generated by two elements ρ and σ such that $\rho^4 = 1$, $\sigma^2 = 1$ and $\sigma\rho\sigma = \rho^{-1}$.

(b) Determine all subgroups of D_8 and describe the action of D_8 on them by conjugation.

(c) Find subgroups $K \lhd H \lhd D_8$ such that K is not normal in D_8.

Problem 8. Let $n \geq 1$. Define a group by generators and relations as

$$D_{2n} = \langle \rho, \sigma \mid \rho^n = \sigma^2 = \sigma\rho\sigma\rho = 1 \rangle.$$

It is called *the dihedral group of order* $2n$.

(a) Show that D_{2n} indeed has order $2n$. (Hint: Embed D_{2n} into $\operatorname{End}_{\mathbf{Ab}}(\mathbb{C})$.)

(b) Identify D_{2n} as the isometries of the regular n-gon (for $n \geq 3$).

(c) Determine the center $Z(D_{2n})$.
(d) Find all normal subgroups of D_{2n}.
(e) Prove that $D_6 \simeq S_3$, but that $D_8 \not\simeq S_4$.

Problem 9. Let $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$ be the subgroup of *inner automorphisms* of G (that is, automorphisms of the form $a \mapsto gag^{-1}$ for some $g \in G$). Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Problem 10.

(a) Let G be a group, and let N be a subgroup of the center $Z(G)$. Show that N is normal in G. Prove that if G/N is cyclic then G is abelian.
(b) Let G be a group and suppose $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian. (Hint: Use the group $\mathrm{Inn}(G)$, defined in Problem 9, and compare with part (a) for N maximal.)

Problem 11. Show that a group with no non-trivial automorphism is trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. (Hint: First check it is abelian and 2-torsion.)