MATH 210A HOMEWORK 2

Problem 1. Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category and $F : \mathcal{I} \to \mathcal{C}$ a functor.

(a) Prove that $\lim_{i\in\mathcal{I}} F(i)$ is characterized by the existence, for every $T \in \mathcal{C}$, of a natural bijection
$$\mathrm{Mor}_{\mathcal{C}}(T, \lim_{i\in\mathcal{I}} F(i)) \simeq \lim_{i\in\mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(T, F(i)).$$

(b) Similarly, $\mathrm{colim}_{i\in\mathcal{I}} F(i)$ is characterized by the existence, for every $T \in \mathcal{C}$, of a natural bijection
$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_{i\in\mathcal{I}} F(i), T) \simeq \lim_{i\in\mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(F(i), T).$$

Problem 2. Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category, and let $\Delta_{\mathcal{I}}(-) : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ denote the constant diagram functor.

(a) Prove that $\mathcal{C}$ admits $\mathcal{I}$-shaped limits if and only if the constant diagram functor $\Delta_{\mathcal{I}}$ admits a right adjoint $\lim_{\mathcal{I}}(-) : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$.

(b) Prove that $\mathcal{C}$ admits $\mathcal{I}$-shaped colimits if and only if the constant diagram functor $\Delta_{\mathcal{I}}$ admits a left adjoint $\mathrm{colim}_{\mathcal{I}}(-) : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$.

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is *continuous* if it preserves all small limits that exist in $\mathcal{C}$ and is *cocontinuous* if it preserves all small colimits that exist in $\mathcal{C}$.

Problem 3. Prove that right adjoints are continuous while left adjoints are cocontinuous.

Problem 4. Let X be a fixed object in a category $\mathcal{C}$.

(a) Consider a new category $\mathcal{C} \downarrow X$ whose objects are the morphisms $f : Y \to X$ (for $Y \in \mathrm{Ob}(\mathcal{C})$) and where a morphism between $f : Y \to X$ and $g : Z \to X$ is a morphism $h : Y \to Z$ in $\mathcal{C}$ such that $g \circ h = f$. Show that $\mathcal{C} \downarrow X$ is a category. This is sometimes called the *comma category of* $\mathcal{C}$ *over* X.

(b) Let $\mathcal{C} \downarrow X$ be as above. Prove that the product of two objects $f : Y \to X$ and $g : Z \to X$ in $\mathcal{C} \downarrow X$ is just the fiber product of Y and Z over X in $\mathcal{C}$. Explicitly describe the fiber product ("pull-back") in the categories **Sets**, **Grps** and **Ab**, if they exist.

Problem 5. Dualize the situation in Problem 4 to define a category $X \downarrow \mathcal{C}$ and the notion of a "push-out." Do push-outs exist in **Sets**, **Grps**, and **Ab**?

Definition. A morphism $f : X \to Y$ is said to be a *monomorphism* if it is *left-cancellable*:
$$(f \circ g_1 = f \circ g_2) \Longrightarrow g_1 = g_2.$$
It is a *split monomorphism* if it is *left-invertible*: there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Dually, an epimorphism is a morphism that is right-cancellable, and a split epimorphism is a morphism that is right-invertible.

Problem 6. Prove that if a morphism is both an epimorphism and a split monomorphism then it is an isomorphism. (Dually, monic and split epi $\Rightarrow$ isomorphism.)

Problem 7. Show that a pull-back of a monomorphism is a monomorphism. More precisely, show that if

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

is a pull-back and f is a monomorphism then h is a monomorphism. (Dually, the push-out of an epimorphism is an epimorphism.)

Problem 8. Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be an adjunction with unit $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$. Prove that the adjunction induces an equivalence between the subcategory

$$\{X \in \mathcal{C} \mid \eta_X \text{ is an isomorphism}\} \subset \mathcal{C}$$

and the subcategory

$$\{Y \in \mathcal{D} \mid \epsilon_Y \text{ is an isomorphism}\} \subset \mathcal{D}.$$

Problem 9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and let G_1 and G_2 be two functors that are right adjoint to F. Prove that G_1 and G_2 are isomorphic. [Hint: Yoneda...]

Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *conservative* if it "reflects isomorphisms": Ff isomorphism $\Rightarrow f$ isomorphism.

Problem 10. Many of the categories $\mathcal{C}$ that we are familiar with (such as **Grps**, **Rings**, **Top**, etc.) possess a "forgetful" functor $U : \mathcal{C} \to$ **Sets**. More generally, there are forgetful functors which don't go all the way down to **Sets**. For example, **Ab** $\to$ **Grps** (forget we are abelian), **Rings** $\to$ **Rngs** (forget we have a unit), and so on.

(a) Explain intuitively why forgetful functors are faithful.
(b) Are forgetful functors conservative in general?

Problem 11. (Free constructions). For many of our familiar categories $\mathcal{C}$, the forgetful functor $U : \mathcal{C} \to$ **Sets** admits a left adjoint $F :$ **Sets** $\to \mathcal{C}$ which sends a set E to the "free object" of the category $\mathcal{C}$ over the set E. The idea might be best illustrated by considering some examples:

(a) Determine the "free commutative ring" over a given set E.
(b) Determine the "free topological space" over a given set E.
(c) Determine the "free ring" over a given rng R.

Problem 12. Does there exist a right adjoint to the forgetful functor $U :$ **Top** $\to$ **Sets**?

Problem 13. Recall the notion of an additive category from Homework 1. A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is *additive* if the induced maps $\mathrm{Mor}_{\mathcal{A}}(X, Y) \to \mathrm{Mor}_{\mathcal{B}}(FX, FY)$ are morphisms of abelian groups.

(a) Prove that a functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is additive iff it preserves biproducts. (Part of the problem is to precisely pin down what this means.)
(b) Give additive versions of the covariant and contravariant Yoneda embeddings using additive functors to **Ab**. What advantage does this additive version have over the usual Yoneda embedding?