MATH 210A HOMEWORK 11

Problem 1. Compute $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ for every $m, n \geq 1$.

Problem 2. Prove that $- \otimes_R -$ is not left exact in either variable.

Problem 3. Verify that $\mathrm{Hom}_R(-, -)$ is left exact in each variable and prove that it is not right exact.

Problem 4. Prove the Snake Lemma for short exact sequences of R-modules.

Problem 5. Let A be a commutative ring and M a finitely generated A-module. Prove that if $f : M \to M$ is A-linear and surjective then it is an isomorphism. (Hint: Nakayama's lemma.)

Problem 6.

(a) Let $R \to S$ be a homomorphism of commutative rings. Let $I \subset R[X_1, \ldots, X_n]$ be an ideal. Show that there is an isomorphism of S-modules (actually of rings, see the next problem):
$$S \otimes_R (R[X_1, \ldots, X_n]/I) \simeq S[X_1, \ldots, X_n]/I \cdot S[X_1, \ldots, X_n]. \tag{1}$$

(b) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$. Compare with $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$.

Problem 7. Let R be a commutative ring and let A and B be two R-algebras (a ring C is an R-*algebra* if there exists a ring homomorphism $R \to C$ whose image is in the center of C; one can then view C as an R-module as usual.) Show that $A \otimes_R B$ becomes a ring such that $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$. Prove for instance that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$ as rings, where $d = \gcd(m, n)$. Show also that the isomorphism (1) in the previous problem is an isomorphism of rings.