MATH 210A HOMEWORK 1

Problem 1. Let X be a topological space. A category $\mathrm{Op}(X)$ may be defined as follows: the objects are the open subsets $U \subset X$ and the morphisms are defined by

$$\mathrm{Mor}(U, V) := \begin{cases} \emptyset & \text{if } U \not\subset V \\ \{\mathrm{incl}_{VU}\} & \text{(one-point set) if } U \subset V. \end{cases}$$

(a) Check that $\mathrm{Op}(X)$ is indeed a category.
(b) For each open subset $U \subset X$, let $\mathcal{F}(U)$ be the set of all continuous functions $U \to \mathbb{R}$. Show that $\mathcal{F}$ defines a contravariant functor from the category $\mathrm{Op}(X)$ to the category of sets. (This is an example of a *presheaf of sets on X*.)

Problem 2. Show that the construction $\mathrm{Op}(-)$ from Problem 1 induces a contravariant functor from the category of topological spaces to the category of small categories.

Problem 3. Does the category $\mathrm{Op}(X)$ have arbitrary products and/or coproducts? If so, describe them.

Problem 4. Determine the initial and final objects in the following categories:

(a) **Sets** of sets,
(b) **FinSets** of finite sets,
(c) **Grps** of groups,
(d) **Ab** of abelian groups,
(e) **Rings** of rings,
(f) **Rngs** of rings without unit,
(g) **Top** of topological spaces.

Problem 5. For the categories of the last problem, discuss the existence of products, coproducts, limits and colimits.

Problem 6. Consider a pair of "parallel" morphisms $f, g : X \to Y$ in a category $\mathcal{C}$. An *equalizer* of f and g is an object Z and a morphism $h : Z \to X$ such that $f \circ h = g \circ h$ and such that for every morphism $i : T \to X$ such that $f \circ i = g \circ i$ there exists a unique morphism $j : T \to Z$ such that $h \circ j = i$. Show that equalizers exist in the category of sets and the category of abelian groups.

Problem 7. Define the dual notion of coequalizer and discuss it in the same examples.

Problem 8. Prove that all limits exist in a category $\mathcal{C}$ if and only if all products and all equalizers exist in $\mathcal{C}$. State and prove the dual result for colimits.

Problem 9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $[\mathcal{C}, \mathcal{D}]$ denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$. Show that if $\mathcal{D}$ is complete (resp. cocomplete) then so is $[\mathcal{C}, \mathcal{D}]$ and that limits (resp. colimits) in $[\mathcal{C}, \mathcal{D}]$ are computed "pointwise."

Problem 10. Define the product $\mathcal{C} \times \mathcal{D}$ of two categories and prove that $[\mathcal{C} \times \mathcal{D}, \mathcal{E}] \simeq [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$. What about $[\mathcal{C}, \mathcal{D} \times \mathcal{E}]$?

Problem 11. Is $[\mathcal{C}, \mathcal{D}]^{\mathrm{op}}$ the same as $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$?

Problem 12. Establish an equivalence between the category of preordered sets (with order-preserving maps as morphisms) and the category of small categories having at most one morphism between any pair of objects.

Problem 13. Convince yourself that a product of an empty collection of objects is a final object, and that the coproduct of an empty collection of objects is an initial object.

Problem 14. A *preadditive category* is a category $\mathcal{A}$ where each morphism set $\mathrm{Mor}_{\mathcal{A}}(X, Y)$ has been equipped with the structure of an abelian group in such a way that composition is a bilinear operation. Note that there is a distinguished morphism $0_{XY} : X \to Y$ between any two objects—namely, the zero element of the abelian group $\mathrm{Mor}_{\mathcal{A}}(X, Y)$. Prove the following elementary facts about preadditive categories:

(a) The composite of a zero morphism with any other morphism is again a zero morphism.
(b) An object is initial if and only if it is final. Thus, the notions of initial, final, and zero object coincide in a preadditive category. (In any category, an object that is both initial and final is called a *zero object*.)
(c) An object Z is a zero object if and only if $\mathrm{id}_Z = 0$.
(d) If a preadditive category has a zero object Z then 0_{XY} is the unique morphism $X \to Y$ which factors through Z.

Problem 15. Let $\mathcal{A}$ be a preadditive category. A *biproduct* of a finite collection of objects $A_1, A_2, \ldots, A_n$ is an object B together with "projection" morphisms $p_k : B \to A_k$ ($k = 1, \ldots, n$) and "injection" morphisms $i_k : A_k \to B$ ($k = 1, \ldots, n$) satisfying $p_k \circ i_k = \mathrm{id}_{A_k}$ for $1 \le k \le n$, $p_k \circ i_j = 0$ for $k \ne j$, and $\mathrm{id}_B = i_1 \circ p_1 + i_2 \circ p_2 + \cdots + i_n \circ p_n$.

(a) Show that $(B, p_1, \ldots, p_n)$ is a product $A_1 \times A_2 \times \cdots \times A_n$ and that $(B, i_1, \ldots, i_n)$ is a coproduct $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$.
(b) Conversely, show that any finite product (resp. coproduct) can be "upgraded" to a biproduct. Conclude that in a preadditive category finite products and finite coproducts coincide. (The $n = 0$ case of this statement was considered in Problem 14(b). Note: a biproduct of an empty collection of objects is conventionally defined to be a zero object.)
(c) An *additive category* is, by definition, a preadditive category with biproducts; equivalently, a preadditive category with finite products; equivalently, a preadditive category with finite coproducts. Convince yourself that these three defintions are indeed equivalent.

Remark: Since finite products and finite coproducts coincide in an additive category, the notation $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ is sometimes used for the product/coproduct/biproduct of $A_1, A_2, \ldots, A_n$. However, infinite products and infinite coproducts (if they exist) will be different in general. Observe that the definition of a biproduct doesn't obviously generalize to an infinite collection of objects.