MATH 210A: HOMEWORK 9

Problem 81. Describe all artinian domains.

Problem 82. Let k be a field of characteristic 0. Define the (non-commutative) Weyl algebra A as the quotient

$$A := k\{X, Y\}/\langle YX - XY - 1\rangle.$$

$k\{X, Y\}$ is not the polynomial algebra $k[X, Y]$ – we mean the free non-commutative algebra on two generators.

(a) Prove that A is simple, that is, it has no two-sided ideals but 0 and A.
(b) Prove that A is not artinian.

The following problems will discuss the completion of a ring at an ideal.

Problem 83. Let $I \subset R$ be a proper left ideal of a (not necessary commutative) ring. We can consider a descending chain of ideals

$$R \supset I \supset I^2 \supset \cdots \supset I^n \supset I^{n+1} \supset \cdots$$

(a) Prove we obtain a sequence

$$\cdots \to R/I^{n+1} \to R/I^n \to \cdots \to R/I^2 \to R/I \to R/R = 0$$

and describe the maps $f_n : R/I^{n+1} \to R/I^n$.

(b) Because the category of rings is complete, we obtain an object

$$\widehat{R}_I := \lim_{n \in \mathbb{N}} (R/I^n)$$

given by the limit of this sequence. This is called the *I-adic completion of R*. Using the canonical maps $\pi_n : R \to R/I^n$, we get a canonical map $\pi : R \to \widehat{R}_I$. What are necessary and sufficient conditions for π to be injective?

Problem 84. Let $I \subset R$ be as above. We now define the *I-adic topology* on R. A subset $U \subset R$ is an open set if and only if for each $x \in U$, there exists $n \in \mathbb{N}$ such that $x + I^n \subset U$.

(a) Prove that open sets are closed under arbitrary union and finite intersection.
(b) What are necessary and sufficient conditions for R to be a Hausdorff space under this topology? Recall that a topological space is Hausdorff if any two points admit disjoint neighborhoods.

Problem 85. If R is given the I-adic topology, then any quotient R/J can inherit the quotient topology.

(a) Prove that R/I^n has the discrete topology for any $n \in \mathbb{N}$.
(b) Prove that the canonical homomorphisms $\pi_n : R \to R/I^n$ are continuous.
(c) Conclude that $\widehat{R}_I$ has a canonical topology. What are the open sets?

Problem 86. A ring R with the I-adic topology admits a notion of Cauchy sequences. A sequence $\{x_i\}$ of elements of R will be called Cauchy if for every $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that for any $i, j > N_k$, $x_i - x_j \in I^k$. Prove that the ring $\widehat{R}_I$ is complete in the I-adic topology it inherits (as discussed above).

Problem 87. Let p be a prime number in $\mathbb{Z}$.

(a) Describe an arbitrary element of $\widehat{\mathbb{Z}}_p := \widehat{\mathbb{Z}}_{p\mathbb{Z}}$.
(b) Are there continuous ring homomorphisms $\widehat{\mathbb{Z}}_p \to \mathbb{R}$?

Problem 88. Find a solution to $x^2 = 10$ in $\widehat{\mathbb{Z}}_3$.

Problem 89. Let R be an integral domain.

(a) Compute the completion of $R[X]$ at the ideal (X), and conclude that it is isomorphic to $R[[X]]$.
(b) What are the continuous ring homomorphisms $\mathbb{Z}[[X]] \to \mathbb{R}$? Hint: there is a universal property for maps between complete rings.

Problem 90. Find a solution to $x^n = 1 - X$ in $\mathbb{Q}[[X]]$.