MATH 210A: HOMEWORK 8

Problem 73. Find an example of a ring that is right noetherian but not left noetherian.

Problem 74. Find an example of a commutative ring that is noetherian but not artinian.

Problem 75. Prove that an artinian integral domain is a field.

Problem 76. Let R be a domain, and $\Sigma \subset R$ a multiplicative set with $0 \notin \Sigma$.

(a) Prove that if R is a PID, then $R[\Sigma^{-1}]$ is still a PID.
(b) If R is a UFD, is $R[\Sigma^{-1}]$ necessarily a UFD?

Problem 77. Let R be a ring, and consider $R[[X]]$, the ring of formal power series with coefficients in R. An element of $R[[X]]$ is

$$f = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n \in R.$$

with multiplication defined by

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i + b_j\right) x^n.$$

(a) Show that $R[[X]]$ is a noetherian if R is.
(b) Show that $f \in R[[X]]$ is invertible if and only if a_0 is.

Problem 78.

(a) Find a Euclidean function $\varphi : \mathbb{Z}[i]\backslash\{0\} \to \mathbb{N}^+$ and prove the Gaussian integers $\mathbb{Z}[i] := \{a + bi : a, b \in \mathbb{Z}\}$ is a Euclidean domain.
(b) Prove that every subring $\mathbb{Z} \subsetneq R \subsetneq \mathbb{Z}[i]$ is not a PID, and in particular cannot be a Euclidean domain.
(c) Prove that a prime number $p \in \mathbb{Z}$ seen as $p \in \mathbb{Z}[i]$ is a prime element if and only if $p \equiv 3 \mod 4$.

Problem 79. Prove that $\mathbb{Z}\left[\sqrt{-5}\right]$ is not a UFD. Hint: find an element which does not admit a unique factorization.

Problem 80. Prove that $R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is not a Euclidean domain. Remark: R is a rare example of a PID that is not a Euclidean domain. This fact is unhelpful for the proof.