MATH 210A: HOMEWORK 7

Problem 63. Find all ring endomorphisms of $\mathbb{R}$, the real numbers. Which are invertible?

Problem 64.

(a) Prove that every two-sided ideal $J \subset M_n(R)$ of the matrix ring is of the form $M_n(I)$ for a two-sided ideal $I \subset R$.
(b) Give an explicit example of a left ideal of $M_n(R)$ that is not of the form $M_n(I)$ for $I \subset R$ a left ideal.

Problem 65. Let R be a commutative ring and $I, J \subset R$ two ideals. Prove that the set of elements $\{ij : i \in I, j \in J\}$ is not necessarily ideal using the ring $\mathbb{Z}[X]$.

Problem 66. For an ideal $I \subset R$ in a commutative ring, we let the *radical* of the ideal be the set

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

(a) Prove that if I is an ideal, so is $\sqrt{I}$.
(b) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
(c) Prove that $\sqrt{\sqrt{I} + \sqrt{J}} = \sqrt{I + J}$.
(d) Give necessary and sufficient conditions for $\sqrt{I} = R$.
(e) Show that $\sqrt{I \cdot J} = \sqrt{I \cap J}$.

Problem 67. For a commutative ring R let $\mathcal{N}(R)$ be the set of nilpotent elements of R, that is all $a \in R$ such that $a^n = 0$ for some $n \in \mathbb{N}$. This is called the nilradical.

(a) Prove that the nilradical is an ideal.
(b) Prove that $\sqrt{\mathcal{N}(R)} = \mathcal{N}(R)$.
(c) Prove that for any ideal $I \subset R$, $\mathcal{N}\left(R/\sqrt{I}\right) = 0$.
(d) Prove that the set of nilpotent elements of $M_2(\mathbb{Z})$ is not an ideal.

Problem 68.

(a) Formalise the statement 'localisation commutes with radicals' and prove it.
(b) Formalise the statement 'localisation commutes with quotients' and prove it.

Problem 69. Let R be a ring, and let $\Sigma \subset R$ be a central multiplicative subset. Let $\varepsilon : R \to R[\Sigma^{-1}]$ be the localisation functor.

(a) What is the kernel of ε?
(b) Show that ε is an epimorphism in the category of rings.
(c) What necessary and sufficient conditions can you put on Σ so that $R[\Sigma^{-1}] \cong 0$?
(d) Show that the association $J \mapsto \varepsilon^{-1}(J)$ defines an injective map φ from the set of one- or two-sided ideals of $R[\Sigma^{-1}]$ to those of R.
(e) What is the image of this map? What is the inverse map from the image of φ to the ideals of $R[\Sigma^{-1}]$?
(f) Formalise the statement 'the quotient rings are preserved under this bijection' and prove it.

Problem 70.

(a) Describe the ideals of $\mathbb{Z}[1/a]$, defined as the localisation of $\mathbb{Z}$ at the multiplicative subset $\{1, a, a^2, \ldots\}$.

(b) Let $p \in \mathbb{Z}$ be a prime number, and let $\Sigma_p = \{a \in \mathbb{Z} : p \nmid a\}$. Prove that Σ_p is multiplicative and describe the ideals of the localisation $\mathbb{Z}_{(p)} := \mathbb{Z}[\Sigma_p^{-1}]$.

Problem 71. Let k be a field, and let $f \in k[X, Y]$ be an irreducible polynomial. Let $\Sigma_f := \{g \in k[X, Y] : f \nmid g\}$.

(a) Prove that Σ_f is a multiplicative set.

(b) Prove that $k[X, Y]_{(f)} := k[X, Y][\Sigma_f^{-1}]$ is a subring of $k(X, Y)$, the field of rational functions of two variables.

(c) Let $r \in k(X, Y)$ be a rational function. Prove that the set

$$\Phi_r := \left\{ f \in k[X, Y] \text{ monic irreducible} : r \notin k[X, Y]_{(f)} \subset k(X, Y) \right\}$$

is finite for each r.

Problem 72. Let R be a ring. An *idempotent* is $e \in R$ such that $e^2 = e$.

(a) Show that if $e \in R$ is a central idempotent, then $R \cong Re \times R(1 - e)$, the left ideals generated by those elements.

(b) Show that if $R = R_1 \times R_2$, there is a central idempotent $e \in R$ such that $R_1 = Re$ and $R_2 = R(1 - e)$. Show also that R_1 and R_2 are localisations of R.

(c) What are the idempotent elements of $\mathbb{Z}/n\mathbb{Z}$?

(d) How many idempotent elements are there in an integral domain?