MATH 210A: HOMEWORK 6

Problem 53. (a) Show that a subgroup of a solvable group remains solvable.

(b) Show that a subgroup of a nilpotent group remains nilpotent.

(c) Show that a quotient of a solvable group remains solvable.

(d) Show that a quotient of a nilpotent group remains nilpotent.

Consider an exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

(e) Show that if N and H are solvable then so is G.

(f) Show that if N and H are nilpotent then G needs not be nilpotent but that it is the case if N is contained in the center of G.

Problem 54. Recall that the derived series for G is defined by $G^{(0)} = G$, and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$, the commutator subgroup of the previous stage.

(a) Prove that the derived series is a normal series of G, i.e. each $G^{(i)}$ is normal in G.

(b) Prove that G is solvable if and only if $G^{(n)}$ is trivial for some $n \in \mathbb{N}$.

Problem 55. Prove the equivalence between being nilpotent and having the ascending, or descending, central series "terminate" (see statement in class).

Problem 56.

(a) Prove that every group of order 12 is solvable.

(b) Using (a), prove that any group of order 588 is solvable.

Problem 57. Find an example of a finite solvable group which admits no *normal* series with each quotient cyclic.

Problem 58. Prove that the dihedral group D_{2n} is solvable for any n.

Problem 59. Prove that the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is solvable.

Problem 60. Prove that the subgroup of $\mathrm{GL}_n(k)$ consisting of upper triangular matrices is solvable for any field k.

Problem 61. Let $G := \langle a, b, c : [a, b] = b, [b, c] = c, [c, a] = a \rangle$, where $[x, y] = xyx^{-1}y^{-1}$. Prove that G is the trivial group.

Problem 62. Show that a finite nilpotent group is the product of its Sylow subgroups.