**Problem 41.** Consider a short exact sequence \( A \to G \to H \) of groups with \( A \) abelian. Construct an action of \( H \) on \( A \) ‘by conjugation’, even if the sequence is not split.

**Problem 42.** Let \( N = S_3 \cong D_6 = \langle x, y \mid x^3 = y^2 = (xy)^2 = 1 \rangle \) and \( G = D_{12} = \langle r, s \mid r^6 = s^2 = (rs)^2 = 1 \rangle \); construct \( N \hookrightarrow G \) and show that \( G/N \cong C_2 =: H \). Show that different sections of \( G \twoheadrightarrow H \) yield different decompositions of \( G \) as \( N \wr H \). Prove that \( D_6 \wr C_2 \cong D_6 \times C_2 \) in a way which is compatible with the short exact sequences \( 1 \to N \to G \to H \to 1 \).

**Problem 43.** Let \( N \) and \( H \) be fixed groups. Show by example that two non-isomorphic actions of \( H \) on \( N \) can give isomorphic semidirect products, compatible with the corresponding short exact sequences \( N \to N \rtimes H \to H \).

**Problem 44.** The Five Lemma for groups:

(a) First, suppose we have a diagram

\[
\begin{array}{c}
A_2 \to A_3 \to A_4 \to A_5 \\
\downarrow f_2 \quad \downarrow f_3 \quad \downarrow f_4 \quad \downarrow f_5 \\
B_2 \to B_3 \to B_4 \to B_5
\end{array}
\]

with exact rows. Prove that if \( f_2 \) and \( f_4 \) are epimorphisms and \( f_5 \) is a monomorphism, then \( f_3 \) is an epimorphism. (In your solution, please label the horizontal arrows as you see fit.)

(b) Second, suppose we have a diagram

\[
\begin{array}{c}
A_1 \to A_2 \to A_3 \to A_4 \\
\downarrow f_1 \quad \downarrow f_2 \quad \downarrow f_3 \quad \downarrow f_4 \\
B_1 \to B_2 \to B_3 \to B_4
\end{array}
\]

with exact rows. Prove that if \( f_2 \) and \( f_4 \) are monomorphisms and \( f_1 \) is an epimorphism, then \( f_3 \) is a monomorphism. (Please make your notation for horizontal arrows consistent.)

(c) State and prove the Five Lemma with the weakest hypotheses necessary.

**Problem 45.**

(a) Let \( p \) be a prime. Prove that every group of order \( p^2 \) is abelian. Hint: Problem 37.

(b) Discuss the types of groups of order \( p^3 \). (The abelian ones are easy to sort out.)

**Problem 46.** Let \( G \) be a finitely-generated group. Prove that \( G \) has only finitely many subgroups of index \( n \) for each \( n \in \mathbb{N} \).

**Problem 47.** Recall that a composition series for \( G \) is a chain of subgroups with \( H_0 = \{e\}, H_n = G, \) and \( H_i \triangleleft H_{i+1} \) such that \( H_{i+1}/H_i \) is simple.

(a) Exhibit all composition series for the dihedral group \( D_8 \).

(b) Exhibit all composition series for the quaternion group \( Q_8 \).
(c) Exhibit all composition series for the symmetric groups $S_n$. Hint: for all but finitely many $n$, the composition series has length 2.

**Problem 48.**

(a) Prove that there are no simple groups of order 132.
(b) Prove that there are no simple groups of order 6545.
(c) Suppose that $G$ is a simple group of order 168. How many elements of order 7 must there be?

**Problem 49.**

(a) Describe all finite groups that have only two conjugacy classes.
(b) Describe all finite groups that have only three conjugacy classes.

**Problem 50.** A group is said to be *nilpotent* if it admits a normal tower $\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ with the property that $H_{i+1}/H_i \subset Z(G/H_i)$ (i.e. is abelian) for each $i$. The minimum possible length of such a “central tower” is called the nilpotency class of $G$.

(a) The upper central series of a group $G$ is a sequence of subgroups defined by setting $Z_0(G) = \{e\}$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G)$ to be the subgroup of $G$ containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. Prove that $G$ is nilpotent if and only if $Z_c(G) = G$ for some $c \in \mathbb{N}$.
(b) The lower central series of a group $G$ is a sequence of subgroups defined by setting $G_0 = G$, $G_1 = [G,G]$, and $G_{i+1} = [G,G_i]$. Prove that $G$ is nilpotent if and only if $G_c = 1$ for some $c \in \mathbb{N}$.
(c) Let $1 \to N \to G \to H \to 1$ be a short exact sequence (i.e. $N \subset Z(G)$). Show that $G$ is nilpotent if and only if $N$ and $H$ are.
(d) Show that any $p$-group is nilpotent.
(e) Show that the cartesian product of a finite number of nilpotent groups is nilpotent.
(f) What is the relationship between nilpotent and solvable groups?

**Problem 51.** Let $G$ be a finite group. Prove that the following are equivalent:

(a) $G$ is nilpotent.
(b) Every Sylow subgroup of $G$ is normal.
(c) $G$ is a direct product of $p$-groups.

**Problem 52.** Find an example of an infinite non-abelian nilpotent group, and prove it is nilpotent. Hint: consider matrix groups.