MATH 210A: HOMEWORK 5

Problem 41. Consider a short exact sequence $A \rightarrowtail G \twoheadrightarrow H$ of groups with A abelian. Construct an action of H on A 'by conjugation', even if the sequence is not split.

Problem 42. Let $N = S_3 \cong D_6 = \langle x, y | x^3 = y^2 = (xy)^2 = 1 \rangle$ and $G = D_{12} = \langle r, s | r^6 = s^2 = (rs)^2 = 1 \rangle$; construct $N \hookrightarrow G$ and show that $G/N \cong C_2 =: H$. Show that different sections of $G \to H$ yield different decompositions of G as $N \rtimes H$. Prove that $D_6 \rtimes C_2 \cong D_6 \times C_2$ in a way which is compatible with the short exact sequences $1 \to N \to G \to H \to 1$.

Problem 43. Let N and H be fixed groups. Show by example that two non-isomorphic actions of H on N can give isomorphic semidirect products, compatible with the corresponding short exact sequences $N \rightarrowtail N \rtimes H \twoheadrightarrow H$.

Problem 44. The Five Lemma for groups:

(a) First, suppose we have a diagram

$$\begin{tikzcd} A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

with exact rows. Prove that if f_2 and f_4 are epimorphisms and f_5 is a monomorphism, then f_3 is an epimorphism. (In your solution, please label the horizontal arrows as you see fit.)

(b) Second, suppose we have a diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_4 \arrow[d, "f_4"'] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \end{tikzcd}$$

with exact rows. Prove that if f_2 and f_4 are monomorphisms and f_1 is an epimorphism, then f_3 is a monomorphism. (Please make your notation for horizontal arrows consistent.)

(c) State and prove the Five Lemma with the weakest hypotheses necessary.

Problem 45.

(a) Let p be a prime. Prove that every group of order p^2 is abelian. Hint: Problem 37.

(b) Discuss the types of groups of order p^3. (The abelian ones are easy to sort out.)

Problem 46. Let G be a finitely-generated group. Prove that G has only finitely many subgroups of index n for each $n \in \mathbb{N}$.

Problem 47. Recall that a composition series for G is a chain of subgroups with $H_0 = \{e\}$, $H_n = G$, and $H_i \lhd H_{i+1}$ such that H_{i+1}/H_i is simple.

(a) Exhibit all composition series for the dihedral group D_8.

(b) Exhibit all composition series for the quaternion group Q_8.

(c) Exhibit all composition series for the symmetric groups S_n. Hint: for all but finitely many n, the composition series has length 2.

Problem 48.

(a) Prove that there are no simple groups of order 132.
(b) Prove that there are no simple groups of order 6545.
(c) Suppose that G is a simple group of order 168. How many elements of order 7 must there be?

Problem 49.

(a) Describe all finite groups that have only two conjugacy classes.
(b) Describe all finite groups that have only three conjugacy classes.

Problem 50. A group is said to be *nilpotent* if it admits a normal tower $\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ with the property that $H_{i+1}/H_i \subset Z(G/H_i)$ (i.e. is abelian) for each i. The minimum possible length of such a "central tower" is called the nilpotency class of G.

(a) The upper central series of a group G is a sequence of subgroups defined by setting $Z_0(G) = \{e\}$, $Z_1(G) = Z(G)$, and $Z_{i+1}(G)$ to be the subgroup of G containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Zi(G))$. Prove that G is nilpotent if and only if $Z_c(G) = G$ for some $c \in \mathbb{N}$.
(b) The lower central series of a group G is a sequence of subgroups defined by setting $G_0 = G$, $G_1 = [G, G]$, and $G_{i+1} = [G, G_i]$. Prove that G is nilpotent if and only if $G_c = 1$ for some $c \in \mathbb{N}$.
(c) Let $1 \to N \to G \to H \to 1$ be a short exact sequence (i.e. $N \subset Z(G)$). Show that G is nilpotent if and only if N and H are.
(d) Show that any p-group is nilpotent.
(e) Show that the cartesian product of a finite number of nilpotent groups is nilpotent.
(f) What is the relationship between nilpotent and solvable groups?

Problem 51. Let G be a finite group. Prove that the following are equivalent:

(a) G is nilpotent.
(b) Every Sylow subgroup of G is normal.
(c) G is a direct product of p-groups.

Problem 52. Find an example of an infinite non-abelian nilpotent group, and prove it is nilpotent. Hint: consider matrix groups.